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ALPHA DECAY OF DEFORMED NUCLEI

BY

PER OLOF FRÖMAN

København 1957
i kommission hos Ejnar Munksgaard

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CONTENTS

Synopsis.

The problem of the alpha fine structure intensities for deformed nuclei is treated on the basis of the unified nuclear model. A system of coupled differential equations for the radial wave functions is obtained; they correspond to the different channels into which the alpha particle is able to leak out, leaving the daughter nucleus in different final states. An approximate analytical formula is derived for the intensities of the different alpha groups in terms of the alpha wave function on the nuclear surface. Also an analysis of the empirical data is presented.

The nuclear deformation causes the alpha decay rate to be enhanced by a factor which increases slowly with the nuclear deformation and which may amount to about 2 for the largest deformations actually found. However, an enhancement factor of this kind is difficult to detect experimentally.

The empirical intensities of the even parity alpha groups of even-even nuclei indicate the existence of higher multipole deformations of even parity in the nuclear surface. The magnitudes of these higher order deformations may change rapidly with the atomic number Z. From the empirical intensities of the odd parity alpha groups which have been found in some even-even nuclei, an estimate is presented of the magnitude of the nuclear octupole moment, the existence of which has been suggested by Christy to explain the presence of the odd parity alpha groups for these even-even nuclei.

The favoured alpha decays of odd-A nuclei and even-even nuclei are similar, and it is possible to obtain an approximate formula which relates the intensities of the favoured alpha groups of an odd-A nucleus to the intensities of the alpha groups of neighbouring even-even nuclei. This intensity formula which was first given in a recent paper by Bohr, Fröman, and Mottelson is discussed in greater detail.

The angular distributions of alpha particles from polarized nuclei are also considered.

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I. Introduction.

A few years after the advent of quantum mechanics, Gamow, Condon, and Gurney explained the puzzling features of alpha decay then known by treating the alpha decay as a barrier penetration problem. (See, e. g. Gamow and Critchfield[28].) In this way, they were able to account approximately for the dependence of the half-life on the alpha particle energy and on the atomic number of the alpha emitting nucleus. Shortly afterwards, Rosenblum discovered the alpha fine structure which is due to the fact that the alpha particle may leave the daughter nucleus in different low-lying states. At that time, one could not account for the details of the fine structure pattern and for the intensities of the alpha groups.

During the last decade, extensive and accurate experimental work has been done in alpha spectroscopy, which has revealed many striking regularities in the alpha fine structure pattern. (See the review article by Perlman and Asaro[43].) These regularities have been found to be related to the rotational level structure which is characteristic of nuclei possessing a shape differing strongly from spherical symmetry. Such large nuclear deformations are known to occur for nuclei with configurations far removed from closed shells, as, for example, in the region of heavy elements ($A > 220$). (See Bohr and Mottelson[16].)

For the interpretation of the alpha fine structure, it is therefore imperative to take into account the departure of the nuclear shape from spherical symmetry. The nuclear deformation is important not only for the energies of the different alpha groups, but also for their relative intensities. Furthermore, as was first pointed out by Hill and Wheeler[33], the nuclear deformation may also affect both the absolute alpha decay rate and the angular distribution of the alpha particles.

In a previous paper by Bohr, Fröman, and Mottelson[15], in which the interpretation of the alpha decay systematics is discussed on the basis of the unified nuclear model, certain simple relations between the relative intensities of the alpha groups are found. In the present paper, the problem of the alpha decay of deformed nuclei is treated in more detail and this treatment is also coordinated with a new analysis of the experimental data. In particular, we derive approximate formulae for the intensities of the different alpha groups in terms of the alpha wave function on the nuclear surface and the shape of this surface. It is found that the intensities depend very sensitively not only on the quadrupole moment, but also on higher multipole moments of the nuclear shape. Thus, the observed alpha intensities may yield information on the details of the nuclear shape as well as on the probability for alpha particle formation in different parts of the nuclear surface.

In Chapter II, we give a brief review of the description of strongly deformed nuclei according to the unified nuclear model. The treatment of the alpha penetration problem is discussed in Chapters III—VI. Chapter VII contains an analysis of the experimental data. The applicability of the theory to the account of empirical data is discussed in Chapter VIII. The paper is followed by two appendices (and a list of references is added). Appendix A contains formulae and curves for the Gamow wave functions which describe the penetration of alpha particles through a spherical Coulomb barrier. Appendix B deals with the penetration of an alpha particle through an anisotropic potential barrier which is fixed in space. This problem is treated by combining a three-dimensional WKB-method, first suggested by CHRISTY, with ideas known from optics.

This work was begun in 1953—1954 when the author was a member of the Theoretical Study Group of CERN (Conseil Européen pour la Recherche Nucléaire) at Universitetets Institut for Teoretisk Fysik in Copenhagen. I wish to thank Professor NIELS BOHR for the privilege of working at his institute. The present investigation was suggested to me by Professor AAGE BOHR, to whom I am deeply indebted for many valuable discussions and advice, and for his continued interest in this work. I am grateful to Dr. BEN MOTTELSON for fruitful discussions of the problem and to Professor I. PERLMAN from the University of California, Berkeley, for numerous data put at my disposal previous to publication. To Professor IVAR WALLER, University of Uppsala, I wish to express my gratitude for what he has taught me during the time when I was his student as well as for his encouragement in the course of this work. Finally, it is a pleasure to thank Mrs. S. HELLMANN for revising the English text and for her untiring help with the correction of the proofs.

II. Survey of Some Features of the Unified Nuclear Model.

The systematic occurrence of rotational nuclear spectra in the region of the very heavy elements shows that it is possible for these nuclei to separate approximately between an intrinsic motion of the nucleons in a deformed field of axial symmetry and a collective rotational motion of the nucleons. Corresponding to this separability of the motion, the normalized nuclear wave functions are*

$$\psi^I_{M,K} = \sqrt{\frac{2I+1}{2}}\left\{\chi_K D^I_{M,K}(\omega) + (-1)^{I-q}\chi_{-K} D^I_{M,-K}(\omega)\right\} \tag{II-1}$$

if $K \neq 0$ and

$$\psi^I_{M,0} = \sqrt{2I+1}\,\chi_0 D^I_{M,0}(\omega) \tag{II-2}$$

if $K = 0$. In these expressions, the functions $\chi_{\pm K}$ which are assumed to form an orthonormal set of functions describe the intrinsic particle and vibrational motion

* The normalization factors in (II-1) and (II-2) differ by a factor $(8\pi^2)^{1/2}$ from the corresponding normalization factors used by BOHR and MOTTELSON[16], owing to the fact that we define the volume element $d\omega$ such that $\int d\omega = 1$, whereas BOHR and MOTTELSON use the definition $\int d\omega = 8\pi^2$.

characterized by the quantum numbers specifying the particle configuration and the excited phonons in the nuclear vibrations. The quantum number K represents the component of the total angular momentum along the nuclear symmetry axis. If nothing else is stated, K is assumed to be non-negative. It seems possible to consider the intrinsic particle motion approximately in terms of the independent motion of the individual nucleons, i. e., to consider the individual particle part of χ_K to be described by a Slater determinant depending on the space and spin coordinates of the individual nucleons relative to a coordinate system which is fixed with respect to the nuclear ellipsoid[41]. For the lowest rotational band of even-even nuclei, the nucleons move in paired orbits, and $K = 0$. In this case, the wave functions are given by (II-2). For odd-A nuclei, odd-odd nuclei, and even-even nuclei with unpaired configurations $K \neq 0$ and the wave functions are given by (II-1). Vibrational oscillations of the nuclear shape may be superimposed on the individual particle motion, but the expected large excitation energy for such vibrations ($\gtrsim 1$ MeV) makes them relatively unimportant for the alpha decay problem. An exception may be the odd parity states which appear in some even-even nuclei and which may represent an excitation of a very soft asymmetric vibration. The symbol q in (II-1) is the sum of two quantities, one depending only on the intrinsic particle state, the other being the number of excited phonons corresponding to spherical harmonics of odd order in the nuclear vibrations. To specify clearly the nuclear vibrational state it is sometimes convenient to use the quantity q as a further index on χ, i. e., to write $\chi_{\pm K, q}$ instead of $\chi_{\pm K}$.

The rotational nuclear motion is described by the D-functions which depend on the Eulerian angles ω defining the orientation of the nucleus. These D-functions are characterized by the quantum numbers I, K, and M which are the total angular momentum of the nucleus, its component along the nuclear axis, and its component along a direction fixed in space (the z-axis), respectively. If the volume element $d\omega$ is defined such that $\int d\omega = 1$, the system of functions $\sqrt{2I+1}\, D^I_{M, \pm K}(\omega)$ is orthonormal. The same is then also true for the system of wave functions $\psi^I_{M, K}$ defined by (II-1) or (II-2).

The energy levels of the nucleus are given as the sum of one term, $T_{\text{part}}(K)$*, which depends on K and on other quantum numbers specifying the intrinsic state, and another term, $T_{\text{rot}}(K, I)$, which depends on both K and I. For a nucleus described by the wave functions (II-1), the possible values of I are

$$I = K, K+1, K+2, \ldots,$$

and the corresponding states of the nucleus have the same parity as the intrinsic state. Furthermore, $T_{\text{rot}}(K, I)$ in this case is given by

$$T_{\text{rot}}(K, I) = \frac{\hbar^2}{2\mathfrak{J}}\left\{ I(I+1) - K(K+1) + a(-1)^{I+\frac{1}{2}}(I+\tfrac{1}{2})\,\delta_{\frac{1}{2}, K}\right\}, \qquad \text{(II-3)}$$

* We note that this term may contain a contribution from vibrational as well as individual particle motion.

where $\mathfrak{J}$ is the effective moment of inertia, and a is the so-called decoupling parameter. For the ground state band of an even-even nucleus described by the wave functions (II-2), $T_{\text{part}}(0) = 0$ and the formula

$$T_{\text{rot}}(0, I) = \frac{\hbar^2}{2\mathfrak{J}} I(I+1) \tag{II-4}$$

applies. The possible values of I are

$$I = 0,\ 2,\ 4, \ldots,$$

and the corresponding states of the nucleus have even parity. Some of the very heavy even-even nuclei appear to have also a low-lying vibrational mode of excitation with $K = 0$ and odd parity. The nuclear wave functions are again given by (II-2), $T_{\text{part}}(0)$ has a magnitude of a few hundred keV, and $T_{\text{rot}}(0, I)$ is given by (II-4) with $\mathfrak{J}$ differing somewhat from its value for the ground state band, and with the spin values

$$I = 1, 3, 5, \ldots .$$

The corresponding states of the nucleus have odd parity.

III. Construction of the Wave Function of the Alpha Penetration Problem.

The quantum numbers of the parent nucleus are recognized by an index i (for initial) and those of the daughter nucleus by an index f (for final). To fulfill automatically the conservation laws for the total angular momentum and for the angular momentum component along the z-axis, we describe the penetration of an alpha particle through the potential barrier by means of the wave function

$$\Psi^{I_i}_{M_i} = e^{-\frac{iEt}{\hbar}} \sum_{K_f} \sum_{l} \sum_{I_f} \frac{1}{r} f_{K_f, l, I_f}(r)\, \Phi^{I_i}_{M_i}(K_f, l, I_f; \omega, \vartheta, \varphi), \tag{III-1}$$

where E is the total energy of the alpha particle and the daughter nucleus in the center-of-mass system, r is the distance between the alpha particle and the center of the daughter nucleus, and (ϑ, φ) are the polar angles of the alpha particle with respect to a center-of-mass coordinate system having axes with directions fixed in space. The Φ-functions appearing in (III-1) are defined by*

$$\Phi^{I_i}_{M_i}(K_f, l, I_f;\ \omega, \vartheta, \varphi) = \sum_{M_f} \sum_{m} (I_f, l; M_f, m \,|\, I_f, l; I_i, M_i)\, \psi^{I_f}_{M_f, K_f} Y_{l, m}(\vartheta, \varphi). \tag{III-2}$$

For the sake of simplicity, the dependence on the variables describing the intrinsic nuclear state has not been explicitly stated in this notation. It is easily seen that the Φ-functions form an orthonormal system of functions. The radial functions $f_{K_f, l, I_f}(r)$

* All spherical harmonics and Clebsch-Gordan coefficients in this paper are defined according to Condon and Shortley[20].

in (III-1) are so far left undetermined. They depend on K_f, which is here used as an abbreviation for all the quantum numbers specifying the intrinsic state, on l and I_f, as well as on K_i and I_i. However, in the notation we have omitted, for the sake of simplicity, explicit reference to the dependence on K_i and I_i. The sum over K_f which should be understood to include also a sum over all the other quantum numbers specifying the individual particle motion and the vibrational motion for the intrinsic state is included on the right-hand side of (III-1) in order to allow the description of transitions to different rotational bands of the daughter nucleus.

The Clebsch-Gordan coefficients in (III-2) are different from zero only if

$$\left| I_i - I_f \right| \leq l \leq I_i + I_f$$

and, therefore, the functions $f_{K_f, l, I_f}(r)$ are defined only if this inequality is fulfilled. This fact is also immediately evident from the rules for the addition of two angular momenta. The requirement of conservation of parity introduces the further restriction that l takes even values if the daughter nucleus is left in a band with the same parity as the parent nucleus, and that l takes odd values if the daughter nucleus is left in a band with a parity opposite to that of the parent nucleus.

So far we have referred the motion of the alpha particle to a center-of-mass coordinate system with axes having directions fixed in space, but this motion may also be referred to a center-of-mass coordinate system with axes having directions fixed with respect to the nuclear ellipsoid. Assuming this latter coordinate system to be a polar one with the polar axis along the axis of the nucleus, and denoting the coordinates of the alpha particle in this coordinate system by $(r, \vartheta', \varphi')$, one realizes that it must be possible to transform the wave function (III-1) into the form

$$\left.\begin{aligned} \Psi^{I_i}_{M_i} = e^{-\frac{iEt}{\hbar}} \sum_{K_f} \sum_{l} \sum_{\Omega} \frac{1}{r} g_{K_f, l, \Omega}(r) \\ \times \sqrt{\frac{2 I_i + 1}{2}} \Big\{ \chi_{K_f} D^{I_i}_{M_i, K_f + \Omega}(\omega)\, Y_{l, \Omega}(\vartheta', \varphi') \\ + (-1)^{I_i - q_f - l} \chi_{-K_f} D^{I_i}_{M_i, -K_f - \Omega}(\omega)\, Y_{l, -\Omega}(\vartheta' \varphi') \end{aligned}\right\} \tag{III-3}$$

if $K_f \neq 0$, and into the form

$$\Psi^{I_i}_{M_i} = e^{-\frac{iEt}{\hbar}} \sum_{q_f} \sum_{l} \sum_{\Omega} \frac{1}{r} g_{0, l, \Omega}(r) \sqrt{2 I_i + 1}\, \chi_0 D^{I_i}_{M_i, \Omega}(\omega)\, Y_{l, \Omega}(\vartheta', \varphi') \tag{III-4}$$

if $K_f = 0$. The functions $\sqrt{\frac{2 I_i + 1}{2}} \{ \quad \}$ in (III-3) and $\sqrt{2 I_i + 1}\, \chi_0 D^{I_i}_{M_i, \Omega}(\omega)\, Y_{l, \Omega}(\vartheta', \varphi')$ in (III-4) form orthonormal systems of functions.

The requirement that the appropriate one of the two wave functions (III-3) and (III-4) shall describe the alpha decay of a parent nucleus having the angular momentum component K_i along its axis shows that, on (or just outside) the nuclear surface, this wave function can contain only angular momentum components Ω fulfilling, $K_f + \Omega = \pm K_i$, i. e., $\Omega = \pm K_i - K_f$. When r is but slightly greater than the nuclear radius, the functions $g_{K_f, l, \Omega}(r)$ can therefore be different from 0 only for $\Omega = \pm K_i - K_f$. For $K_i = 0$, there exists only one such function if $l \geq K_f$, but none if $l < K_f$. For $K_i \neq 0$, there exist two if $l > K_i + K_f$, one if $|K_i - K_f| \leq l < K_i + K_f$, but none if $l < |K_i - K_f|$. Just outside the nuclear surface, the functions $g_{K_f, l, \Omega}(r)$ are therefore different from zero for at most two Ω-values, but, for larger values of r, these functions may be different from zero for all possible Ω-values, since in general Ω is not a constant of the motion during the penetration (except for the decay of even-even nuclei).

The derivation of the connection between the functions $f_{K_f, l, I_f}(r)$ and $g_{K_f, l, \Omega}(r)$, which we find in (III-9), requires some calculations. These calculations as well as those in the following chapter will be done for the case $K_f \neq 0$, i. e., assuming that the wave functions of the daughter nucleus are given by (II-1). However, in reality this is no restriction, for from (II-1) and (II-2) it is clear that the formulae for the case $K_f = 0$ can be obtained from the formulae for the case $K_f \neq 0$ by leaving out the factor $1/\sqrt{2}$ and the second term in the brackets $\{\quad\}$, and by putting $K_f = 0$.

By substituting (II-1) into (III-2), one finds

$$\left.\begin{aligned} &\Phi^{I_i}_{M_i}(K_f, l, I_f;\ \omega, \vartheta, \varphi) \\ &= \frac{1}{\sqrt{2}}\left\{\chi_{K_f}\varphi^{I_i}_{M_i}(K_f, l, I_f;\ \omega, \vartheta, \varphi) + (-1)^{I_f - q_f}\chi_{-K_f}\varphi^{I_i}_{M_i}(-K_f, l, I_f;\ \omega, \vartheta, \varphi)\right\}, \end{aligned}\right\} \quad \text{(III-5)}$$

$\varphi^{I_i}_{M_i}$ being defined by*

$$\left.\begin{aligned} &\varphi^{I_i}_{M_i}(K_f, l, I_f;\ \omega, \vartheta, \varphi) \\ &= \sum_{M_f}\sum_{m}\left(I_f, l;\ M_f, m \,|\, I_f, l;\ I_i, M_i\right)\sqrt{2 I_f + 1}\, D^{I_f}_{M_f, K_f}(\omega)\, Y_{l, m}(\vartheta, \varphi). \end{aligned}\right\} \quad \text{(III-6)}$$

Using the formulae*

$$Y_{l, m}(\vartheta, \varphi) = \sum_{\Omega} D^{l}_{m, \Omega}(\omega)\, Y_{l, \Omega}(\vartheta', \varphi'),$$

$$\begin{aligned} &D^{I_f}_{M_f, K_f}(\omega)\, D^{l}_{m, \Omega}(\omega) \\ &= \sum_{k}(I_f, l;\ M_f, m \,|\, I_f, l;\ k, M_f + m)(I_f, l;\ K_f, \Omega \,|\, I_f, l;\ k, K_f + \Omega)\, D^{k}_{M_f + m, K_f + \Omega}(\omega) \end{aligned}$$

and

$$\sqrt{2 I_f + 1}\,(I_f, l;\ K_f, \Omega \,|\, I_f, l;\ k, K_f + \Omega) = (-1)^{I_f - k + \Omega}\sqrt{2k + 1}\,(k, l;\ K_f + \Omega, -\Omega \,|\, k, l;\ I_f, K_f),$$

one easily finds that (III-6) becomes*

* In (III-6) and (III-7), as well as in the formulae between (III-6) and (III-7), the quantum number K_f is allowed to take also negative values.

$$\left.\begin{aligned}\varphi^{I_i}_{M_i}(K_f, l, I_f;\ \omega, \vartheta, \varphi) = \sum_{\Omega} (-1)^{I_f - I_i + \Omega} (I_i, l; K_f + \Omega, -\Omega \,|\, I_i, l; I_f, K_f) \\ \times \sqrt{2 I_i + 1}\, D^{I_i}_{M_i, K_f + \Omega}(\omega)\, Y_{l, \Omega}(\vartheta', \varphi').\end{aligned}\right\} \quad \text{(III-7)}$$

From (III-5), (III-7), and the formula

$$(I_i, l; -K_f - \Omega, \Omega \,|\, I_i, l; I_f, -K_f) = (-1)^{I_i - l - I_f + 2\Omega} (I_i, l; K_f + \Omega, -\Omega \,|\, I_i, l; I_f, K_f)$$

we then find that

$$\left.\begin{aligned}\Phi^{I_i}_{M_i}(K_f, l, I_f;\ \omega, \vartheta, \varphi) = \sum_{\Omega} (-1)^{I_f - I_i + \Omega} (I_i, l;\ K_f + \Omega, -\Omega \,|\, I_i, l;\ I_f, K_f) \\ \times \sqrt{\frac{2I_i + 1}{2}} \Big\{ \chi_{K_f} D^{I_i}_{M_i, K_f + \Omega}(\omega)\, Y_{l, \Omega}(\vartheta', \varphi') \\ + (-1)^{I_i - q_f - l} \chi_{-K_f} D^{I_i}_{M_i, -K_f - \Omega}(\omega)\, Y_{l, -\Omega}(\vartheta', \varphi') \Big\}.\end{aligned}\right\} \quad \text{(III-8)}$$

This formula and the corresponding formula for the case $K_f = 0$ show that the wave function (III-1) can always be written in one of the forms (III-3) or (III-4) and that the connection between the functions $f_{K_f, l, I_f}(r)$ and $g_{K_f, l, \Omega}(r)$ is

$$g_{K_f, l, \Omega}(r) = \sum_{I_f} (-1)^{I_f - I_i + \Omega} (I_i, l;\ K_f + \Omega,\ -\Omega \,|\, I_i, l;\ I_f, K_f)\, f_{K_f, l, I_f}(r) \quad \text{(III-9)}$$

which can also be written as follows:

$$f_{K_f, l, I_f}(r) = \sum_{\Omega} (-1)^{I_f - I_i + \Omega} (I_i, l;\ K_f + \Omega, -\Omega \,|\, I_i, l;\ I_f, K_f)\, g_{K_f, l, \Omega}(r). \quad \text{(III-10)}$$

For the alpha groups of even-even nuclei ($I_i = K_f = 0$), these two formulae become

$$g_{0, l, \Omega}(r) = (-1)^l f_{0, l, l}(r)\, \delta_{0, \Omega} \quad \text{(III-11)}$$

$$f_{0, l, I_f}(r) = (-1)^l g_{0, l, 0}(r)\, \delta_{l, I_f}. \quad \text{(III-12)}$$

IV. Partial Transition Probabilities and Angular Distributions of the Alpha Groups.

By an alpha group we denote those alpha particles which correspond to a transition from a certain state (K_i, I_i) of the parent nucleus to a certain state (K_f, I_f) of the daughter nucleus. It is the purpose of this chapter to give the formulae relating the partial transition probabilities and the angular distributions of the alpha groups to the asymptotic values (for large values of r) of the functions $f_{K_f, l, I_f}(r)$ which appear in the wave function discussed in the previous chapter. It is assumed that the functions $f_{K_f, l, I_f}(r)$ are normalized such that they correspond to the alpha decay of a single atomic nucleus. Furthermore, we consider the total energy E as real, thus avoiding the unessential complication with complex eigenvalues and wave functions tending to infinity far away from the nucleus.

From the wave function (III-1), which is valid both for $K_f \neq 0$ and for $K_f = 0$, one immediately finds that the probability per unit time, that an alpha particle leaks out with the angular momentum l leaving the daughter nucleus in a state (K_f, I_f), is

$$P_{K_f, l, I_f} = \lim_{r \to \infty} v \left| f_{K_f, l, I_f}(r) \right|^2, \qquad \text{(IV-1)}$$

l taking either only even values or only odd values depending on the parities of the parent and daughter nuclei. The velocity of the alpha particle relative to the daughter nucleus is here denoted by v. The sum of the kinetic energy of the alpha particle and the kinetic recoil energy of the daughter nucleus is called the transition energy, and is denoted by E_{trans}. The quantities E_{trans} and v are connected by the formula

$$E_{\text{trans}} = \tfrac{1}{2} m v^2,$$

where m is the reduced mass of the alpha particle and the daughter nucleus.

The partial half-life T_{K_f, l, I_f} which corresponds to the partial transition probability P_{K_f, l, I_f} is

$$T_{K_f, l, I_f} = \frac{\ln 2}{P_{K_f, l, I_f}}.$$

From (III-1) and (III-2) one easily finds that the probability per unit time and unit solid angle (in the center-of-mass system), that an alpha particle leaks out in the direction (ϑ, φ) leaving the daughter nucleus in the state (K_f, I_f, M_f), is

$$\lim_{r \to \infty} \frac{1}{2\pi} v \left| \sum_l (I_f, l; M_f, M_i - M_f \,|\, I_f, l; I_i, M_i) \, f_{K_f, l, I_f}(r) \, \Theta_{l, M_i - M_f}(\vartheta) \right|^2, \qquad \text{(IV-2)}$$

where $\Theta_{l, m}(\vartheta)$ denotes the normalized ϑ-depending factor of the spherical harmonic $Y_{l, m}(\vartheta, \varphi)$. (See the book on the Theory of Atomic Spectra by Condon and Shortley[20].) By summing the expression (IV-2) over the possible M_f-values and averaging it with respect to the initial distribution of M_i-values, we can calculate the angular distributions of the different alpha groups emerging from polarized parent nuclei.

V. Differential Equations of the Alpha Penetration Problem.

In the center-of-mass system of the alpha particle and the daughter nucleus, the Hamiltonian of this two-body system is

$$H = -\frac{\hbar^2}{2m} \Delta_{\mathbf{r}} + V(\mathbf{r}') + T_{\text{part}} + T_{\text{rot}}. \qquad \text{(V-1)}$$

The reduced mass of the alpha particle and the daughter nucleus is denoted by m, the Laplacian with respect to $\boldsymbol{r}$ by $\Delta_{\boldsymbol{r}}$, the interaction energy between the alpha particle and the daughter nucleus by $V(\boldsymbol{r}')$, and the operator for the excitation energy of the daughter nucleus by $T_{\text{part}} + T_{\text{rot}}$. The vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$ are position vectors for the alpha particle relative to two different coordinate systems. The vector $\boldsymbol{r} = (r, \vartheta, \varphi)$ is referred to a center-of-mass coordinate system whose axes have fixed directions in space, whereas the vector $\boldsymbol{r}' = (r, \vartheta', \varphi')$ is referred to a center-of-mass coordinate system whose axes have fixed directions with respect to the nuclear ellipsoid of the daughter nucleus.

The wave function $\Psi^{I_i}_{M_i}$ which describes the barrier penetration of an alpha particle shall satisfy the wave equation

$$\left(H + \frac{\hbar}{i}\frac{\partial}{\partial t}\right)\Psi^{I_i}_{M_i} = 0 \tag{V-2}$$

and the boundary conditions that the alpha wave function is assumed to be known on the nuclear surface and that it shall represent only outgoing waves for large values of r.

Starting from (V-1) and (V-2) we derive, in this chapter, differential equations for the functions $f_{K_f, l, I_f}(r)$ and $g_{K_f, l, \Omega}(r)$, which we have introduced in Chapter III. Although the calculations are here based on the wave function (III-3), i. e., we assume that $K_f \neq 0$, it is easily seen that the resulting differential equations for the functions $f_{K_f, l, I_f}(r)$ and $g_{K_f, l, \Omega}(r)$ are valid also for $K_f = 0$.

For the Laplacian $\Delta_{\boldsymbol{r}}$ we have the well-known formula

$$\Delta_{\boldsymbol{r}} = \frac{1}{r}\frac{\partial^2}{\partial r^2} r - \frac{\boldsymbol{L}^2}{r^2},$$

where $\boldsymbol{L}$ is the orbital angular momentum (in units of $\hbar$). Expressed as a differential operator in the polar angles, $\boldsymbol{L}^2$ has, of course, the same form whether it is expressed in the angles (ϑ, φ) or in the angles (ϑ', φ'). From (III-3) and the above formula for $\Delta_{\boldsymbol{r}}$ we get

$$\left.\begin{aligned} \Delta_{\boldsymbol{r}} \Psi^{I_i}_{M_i} = e^{-\frac{iEt}{\hbar}} \sum_{K_f}\sum_{l}\sum_{\Omega} \frac{1}{r}\left(\frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2}\right) g_{K_f, l, \Omega}(r) \\ \times \sqrt{\frac{2 I_i + 1}{2}} \Big\{ \chi_{K_f} D^{I_i}_{M_i, K_f + \Omega}(\omega)\, Y_{l, \Omega}(\vartheta', \varphi') \\ + (-1)^{I_i - q_f - l} \chi_{-K_f} D^{I_i}_{M_i, -K_f - \Omega}(\omega)\, Y_{l, -\Omega}(\vartheta', \varphi') \Big\}. \end{aligned}\right\} \tag{V-3}$$

The axially symmetric potential $V(\boldsymbol{r}')$ which depends on the vector $\boldsymbol{r}'$ and on the vibrational coordinates for the nucleus can be expanded as follows:

$$V(\boldsymbol{r}') = \sum_{\lambda = 0}^{\infty} V_\lambda(r)\, Y_{\lambda, 0}(\vartheta'), \tag{V-4}$$

where the $V_\lambda(r)$ are expansion coefficients depending on r and the vibrational coordinates. It is convenient to introduce a function $U_0(r)$ which represents essentially the spherically symmetric part of $V(\boldsymbol{r}')$, but is slightly modified so as to be independent of the nuclear vibrations. The function $V(\boldsymbol{r}')\,\chi_{K_f, q_f} Y_{l,\Omega}(\vartheta', \varphi')$ can then be expanded as follows:

$$\left.\begin{aligned} & V(\boldsymbol{r}')\,\chi_{K_f, q_f} Y_{l,\Omega}(\vartheta', \varphi') \\ & = U_0(r)\,\chi_{K_f, q_f} Y_{l,\Omega}(\vartheta', \varphi') + \sum_{q_f'}\sum_{l'} \langle q_f' \,|\, V^{\Omega}_{l',l}(r) \,|\, q_f \rangle\, \chi_{K_f, q_f'} Y_{l',\Omega}(\vartheta', \varphi'), \end{aligned}\right\} \tag{V-5}$$

where

$$\left.\begin{aligned} V^{\Omega}_{l',l}(r) = \sum_{\lambda} (-1)^{\lambda} \sqrt{\frac{2\lambda+1}{4\pi}}\, & (l, \lambda; -\Omega, 0 \,|\, l, \lambda; l', -\Omega) \\ \times (l', \lambda;\ 0, 0 \,|\, l', \lambda;\ l, 0) & \left\{ V_\lambda(r) - U_0 \sqrt{4\pi}\,\delta_{\lambda,0} \right\} \end{aligned}\right\} \tag{V-6}$$

according to (V-4) and the fact that the integral (over all directions) of the product of three spherical harmonics can be expressed in terms of Clebsch-Gordan coefficients. If r is larger than the largest axis of the nuclear ellipsoid, the term in (V-6) which corresponds to $\lambda = 2$ can be written

$$(l, 2; -\Omega, 0 \,|\, l, 2;\ l', -\Omega)\,(l', 2; 0, 0 \,|\, l', 2;\ l, 0)\,\frac{\varepsilon^2 Q_0}{r^3}, \tag{V-7}$$

as is easily seen from the formulae (V-10) and (V-13) which will be given later. The quantities $\langle q_f' \,|\, V^{\Omega}_{l',l}(r) \,|\, q_f \rangle$ appearing in (V-5) are identically equal to zero, unless $(q_f' + l') - (q_f + l)$ is even and both l and l' are $\geq |\Omega|$, and they also have the following symmetry properties:

$$\langle q_f' \,|\, V^{\Omega}_{l',l}(r) \,|\, q_f \rangle = \langle q_f \,|\, V^{\Omega}_{l,l'}(r) \,|\, q_f' \rangle = \langle q_f \,|\, V^{-\Omega}_{l,l'}(r) \,|\, q_f' \rangle .$$

From (III-3) and (V-5) we get

$$\left.\begin{aligned} V(\boldsymbol{r}')\,\Psi^{I_i}_{M_i} = e^{-\frac{iEt}{\hbar}} \sum_{K_f}\sum_{l}\sum_{\Omega} \frac{1}{r} \Big(U_0(r)\, g_{K_f, l, \Omega}(r) \\ + \sum_{q_f'}\sum_{l'} \langle q_f \,|\, V^{\Omega}_{l,l'}(r) \,|\, q_f' \rangle\, g_{K_f', l', \Omega}(r) \Big) \sqrt{\frac{2I_i+1}{2}} \Big\{ \chi_{K_f} D^{I_i}_{M_i, K_f+\Omega}(\omega)\, Y_{l,\Omega}(\vartheta', \varphi') \\ + (-1)^{I_i - q_f - l}\, \chi_{-K_f} D^{I_i}_{M_i, -K_f-\Omega}(\omega)\, Y_{l,-\Omega}(\vartheta', \varphi') \Big\} \end{aligned}\right\} \tag{V-8}$$

if, in the symmetrization term, use is also made of the fact that $\langle q_f \,|\, V^{\Omega}_{l,l'}(r) \,|\, q_f' \rangle$ is independent of the sign of Ω and is zero unless $(q_f + l) - (q_f' + l')$ is even. It is tacitly implied that the sum over K_f includes also a sum over the vibrational quantum number q_f on which $g_{K_f, l, \Omega}(r)$ depends, although this dependence is not indicated explicitly in the notation.

The potential $V(\boldsymbol{r}')$ is strongly influenced by the nuclear forces when the alpha particle is inside the nucleus*, but it is due only to the electrostatic forces when the alpha particle is outside the surface of the daughter nucleus, and then the potential is given by

$$V(\boldsymbol{r}') = \int \frac{2\,\varepsilon\,\varrho(\boldsymbol{r}'')}{|\boldsymbol{r}'-\boldsymbol{r}''|}\,d\boldsymbol{r}'', \tag{V-9}$$

where ε is the absolute value of the electron charge, and $\varrho(\boldsymbol{r}'')$ is the charge density of the daughter nucleus at the point characterized by the position vector $\boldsymbol{r}''$ which (like $\boldsymbol{r}'$) is referred to the coordinate system having axes whose directions are fixed with respect to the nuclear ellipsoid. From (V-9) we find that, if r is larger than the largest axis of the nuclear ellipsoid, the coefficients $V_\lambda(r)$, defined by (V-4), are given by

$$V_\lambda(r) = \frac{8\,\pi\,\varepsilon\,Q'_\lambda}{(2\lambda+1)\,r^{\lambda+1}}, \tag{V-10}$$

where

$$Q'_\lambda = \int r^\lambda\,Y_{\lambda,0}(\vartheta')\,\varrho(\boldsymbol{r}')\,d\boldsymbol{r}'. \tag{V-11}$$

As special cases of this formula, one has

$$Q'_0 = \frac{(Z-2)\,\varepsilon}{\sqrt{4\,\pi}}, \tag{V-12}$$

since the total charge of the daughter nucleus is $(Z-2)\,\varepsilon$, and

$$Q'_2 = \frac{1}{2}\sqrt{\frac{5}{4\,\pi}}\,\varepsilon\,Q_0, \tag{V-13}$$

where

$$\varepsilon\,Q_0 = \int r^2\,2\,P_2(\cos\vartheta')\,\varrho(\boldsymbol{r}')\,d\boldsymbol{r}', \tag{V-14}$$

Q_0 being the intrinsic quadrupole moment of the daughter nucleus.

If the nuclear surface is represented by

$$R(\vartheta') = R_0\left(1 + \sum_{\lambda>0}\beta_\lambda\,Y_{\lambda,0}(\vartheta')\right) \tag{V-15}$$

and the nucleus is assumed to be uniformly charged, one easily finds that

$$Q'_\lambda = \frac{3}{4\,\pi}(Z-2)\,\varepsilon\,R_0^\lambda\,\beta_\lambda \qquad (\lambda>0), \tag{V-16}$$

provided that the deformation parameters $\beta_\lambda\ (\lambda>0)$ are very small. In more general cases, Q'_λ depends not only on β_λ, but also on the other parameters $\beta_{\lambda'}\ (\lambda' \neq \lambda)$.

The wave function $\psi^{I_f}_{M_f,\,K_f}$ of the daughter nucleus is an eigenfunction of the operators T_{part} and T_{rot}, the corresponding eigenvalues being $T_{\text{part}}(K_f)$ and

* Inside the nucleus, the usefulness of the simple two-body potential $V(\boldsymbol{r}')$ may be restricted to the region near the nuclear surface.

$T_{\mathrm{rot}}(K_f, I_f)$ respectively. From (III-1), (III-2), (III-8), (III-9), and (III-10) we therefore get

$$\left.\begin{aligned}&(T_{\mathrm{part}} + T_{\mathrm{rot}})\,\Psi^{I_i}_{M_i}\\ &= e^{-\frac{iEt}{\hbar}} \sum_{K_f}\sum_{l}\sum_{\Omega}\frac{1}{r}\left[T_{\mathrm{part}}(K_f)\, g_{K_f,l,\Omega}(r) + \sum_{\Omega'} A^{I_i}_{K_f,l,\Omega,\Omega'}\, g_{K_f,l,\Omega'}(r)\right]\\ &\times\sqrt{\frac{2I_i+1}{2}}\Big\{\chi_{K_f} D^{I_i}_{M_i,K_f+\Omega}(\omega)\, Y_{l,\Omega}(\vartheta',\varphi')\\ &+(-1)^{I_i-q_f-l}\chi_{-K_f} D^{I_i}_{M_i,-K_f-\Omega}(\omega)\, Y_{l,-\Omega}(\vartheta',\varphi')\Big\},\end{aligned}\right\}\qquad \text{(V-17)}$$

where

$$\left.\begin{aligned}&A^{I_i}_{K_f,l,\Omega,\Omega'} = (-1)^{\Omega-\Omega'}\sum_{I_f} T_{\mathrm{rot}}(K_f, I_f)\\ &\times(I_i, l;\, K_f+\Omega, -\Omega \,|\, I_i, l;\, I_f, K_f)\,(I_i, l;\, K_f+\Omega', -\Omega' \,|\, I_i, l;\, I_f, K_f).\end{aligned}\right\}\qquad \text{(V-18)}$$

For the alpha groups of even-even nuclei ($I_i = K_f = 0$), this formula becomes

$$A^{0}_{0,l,\Omega,\Omega'} = \frac{\hbar^2\, l\,(l+1)}{2\,\mathfrak{J}}\,\delta_{0,\Omega}\,\delta_{0,\Omega'}. \qquad \text{(V-19)}$$

From (III-3), (V-1), (V-2), (V-3), (V-8), and (V-17) we now get the following system of coupled differential equations:

$$\left.\begin{aligned}&\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + U_l(r) - E_{K_f}\right) g_{K_f,l,\Omega}(r)\\ &+\sum_{q'_f}\sum_{l'}\langle q_f |\, V^{\Omega}_{l,l'}(r)\, | q'_f\rangle\, g_{K'_f,l',\Omega}(r) + \sum_{\Omega'} A^{I_i}_{K_f,l,\Omega,\Omega'}\, g_{K_f,l,\Omega'}(r) = 0,\end{aligned}\right\}\qquad \text{(V-20)}$$

where

$$U_l(r) = U_0(r) + \frac{\hbar^2\, l\,(l+1)}{2\,m\,r^2} \qquad \text{(V-21)}$$

and

$$E_{K_f} = E - T_{\mathrm{part}}(K_f). \qquad \text{(V-22)}$$

In the interior part of the Coulomb barrier, we can in first approximation neglect the last term in (V-20), describing the rotational energy of the daughter nucleus. To some extent, we can compensate for this approximation by replacing E_{K_f} in the first term of (V-20) by $E_{K_f} - \Delta E_{K_f}$, where ΔE_{K_f} is a constant which is to be chosen suitably. Then, (V-20) is replaced by

$$\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + U_l(r) - (E_{K_f} - \Delta E_{K_f})\right\} g_{K_f,l,\Omega}(r) + \sum_{l'}{}' V^{\Omega}_{l,l'}(r)\, g_{K_f,l',\Omega}(r) = 0 \qquad \text{(V-23)}$$

if we neglect the vibrations and the possibility for transitions between different bands of the daughter nucleus during the alpha particle penetration through the potential barrier. This system of coupled differential equations is equivalent to the Schrödinger equation

$$\left\{-\frac{\hbar^2}{2\,m}\Delta_{\boldsymbol{r}'}+V(\boldsymbol{r}')-(E_{K_f}-\Delta E_{K_f})\right\}\psi=0\,, \tag{V-24}$$

where

$$\psi=\frac{1}{r}\sum_{l}\sum_{\Omega}g_{K_f,l,\Omega}(r)\,Y_{l,\Omega}(\vartheta',\varphi'). \tag{V-25}$$

For those alpha groups of even-even nuclei which emerge from the parent nucleus in its ground state $0+$, and which leave the daughter nucleus in its lowest rotational band ($I_i=K_f=0$), the differential equations (V-20) become

$$\left\{-\frac{\hbar^2}{2\,m}\frac{d^2}{dr^2}+U_l(r)-\left(E-\frac{\hbar^2\,l(l+1)}{2\,\mathfrak{J}}\right)\right\}g_{0,l,0}(r)+\sum_{l'}V^0_{l,l'}(r)\,g_{0,l',0}(r)=0\,, \tag{V-26}$$

the functions $g_{0,l,\Omega}(r)$ being identically equal to zero, unless $\Omega=0$.

So far, we have derived the differential equations for the functions $g_{K_f,l,\Omega}(r)$, but we need also the differential equations for the functions $f_{K_f,l,I_f}(r)$. These latter differential equations can be derived either by starting from (III-1) and proceeding in a similar way as when deriving the former differential equations, or they can be derived from (V-20) by using (III-9) and (V-18). The result is

$$\left.\begin{aligned}&\left(-\frac{\hbar^2}{2\,m}\frac{d^2}{dr^2}+U_l(r)-E_{K_f,I_f}\right)f_{K_f,l,I_f}(r)\\&+\sum_{q_f}\sum_{l'}\sum_{I_f}(-1)^{I_f-I'_f}\sum_{\Omega}(I_i,l;\,K_f+\Omega,-\Omega\,|\,I_i,l;\,I_f,K_f)\\&\times\langle q_f\,|\,V^{\Omega}_{l,l'}(r)\,|\,q'_f\rangle\,(I_i,l';\,K_f+\Omega,-\Omega\,|\,I_i,l';\,I'_f,K_f)\,f_{K'_f,l',I'_f}(r)=0\,,\end{aligned}\right\} \tag{V-27}$$

where

$$E_{K_f,I_f}=E-T_{\text{part}}(K_f)-T_{\text{rot}}(K_f,I_f). \tag{V-28}$$

Obviously, E_{K_f,I_f} is equal to the sum of the kinetic energy of the alpha particle and the recoil energy of the daughter nucleus and, hence, no divergency exists between E_{K_f,I_f} and the transition energy E_{trans} mentioned in the previous chapter.

In the outer region of the $\boldsymbol{r}$-space where the deviation of $V(\boldsymbol{r}')$ from spherical symmetry can be neglected, (V-27) simplifies to

$$\left(-\frac{\hbar^2}{2\,m}\frac{d^2}{dr^2}+U_l(r)-E_{K_f,I_f}\right)f_{K_f,l,I_f}(r)=0\,. \tag{V-29}$$

For the alpha groups corresponding to transitions from the ground state $0+$ of an even-even parent nucleus to the lowest rotational band of the daughter nucleus

$(I_i = K_f = 0)$, Eq. (V-27) becomes

$$\left.\begin{aligned}&\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+U_l(r)-\left(E-\frac{\hbar^2 l(l+1)}{2\mathfrak{J}}\right)\right\}f_{0,l,l}(r)\\&\qquad+\sum_{l'}(-1)^{l-l'}V^0_{l,l'}(r)\,f_{0,l',l'}(r)=0,\end{aligned}\right\}\qquad \text{(V-30)}$$

the functions $f_{0,l,I_f}(r)$ being equal to zero, unless $l = I_f$. From (III-11) and (III-12) it is immediately seen that the differential equations (V-26) and (V-30) are equivalent.

The differential equations for the alpha decay of deformed nuclei have also been considered by RASMUSSEN[44], by RASMUSSEN and SEGALL[45], and by STRUTINSKY[53a].

VI. Approximate Solution of the Alpha Penetration Problem.

In the region just outside the nuclear surface, where appreciable exchange of angular momentum takes place between the alpha particle and the daughter nucleus, the term $T_{\rm rot}$ in the Hamiltonian is relatively unimportant, and we shall neglect it in this region*. Neglecting also transitions between different rotational bands of the daughter nucleus, we can then in this interior region use the simple three-dimensional differential equation** (V-24), the solution of which gives $g_{K_f,l,\Omega}(r)$ according to (V-25) and $f_{K_f,l,I_f}(r)$ according to (III-10). On the other hand, we know that, for large values of r the potential $V(\boldsymbol{r}')$ is nearly spherically symmetric, so that we can use the differential equation (V-29) for $f_{K_f,l,I_f}(r)$. In this outer region, the function $f_{K_f,l,I_f}(r)$ is therefore approximately equal to a solution of (V-29) representing an outgoing wave for large values of r. Such a solution is uniquely determined, except for an arbitrary factor which does not depend on r. With a convenient normalization of this constant factor, we denote the required solution of (V-29) by $G_l(E_{I_f}, r)$ (cf. Appendix A). To obtain an approximate solution of the alpha penetration problem we assume that the interior region where (V-24) can be used, and the exterior region where (V-29) can be used, overlap. In the common part of these two regions, we imagine a spherical surface whose center coincides with that of the nucleus. This sphere of radius R_1 may be considered as the boundary between the two regions where the above mentioned two different approximation procedures apply. The value of the radius R_1 does not appear in the final formulae, but one should imagine R_1 to be chosen such that the approximations introduced can be justified as well as

*) For the justification of our approximation procedure it would have been somewhat better to replace $T_{\rm rot}$ by a suitably chosen constant ΔE_{K_f} in the interior region instead of neglecting it there. (See (V-23) or (V-24) and also the footnote on p. 20.) The value of B, defined by (VI-9) and appearing in the final formulae, would slightly depend on the value chosen for the constant ΔE_{K_f}. This slight dependence is, however, unimportant.

** As has been explained in the previous footnote, the constant ΔE_{K_f} appearing in (V-24) is chosen equal to zero.

possible. To this purpose, it seems appropriate to choose R_1 such that the above mentioned spherical surface lies approximately in the middle of the potential barrier.

The surface of the daughter nucleus is assumed to be represented by*

$$R(\vartheta') = R_0 \sum_{\lambda=0}^{\infty} \beta_\lambda Y_{\lambda,0}(\vartheta'), \tag{VI-1}$$

where $\beta_0 = \sqrt{4\pi}$, $\beta_1 = 0$ and $|\beta_\lambda| << |\beta_2|$ for $\lambda > 2$. For small values of β_2, the intrinsic quadrupole moment Q_0 and the deformation parameter β_2 are connected according to the formula**

$$Q_0 = \frac{3}{\sqrt{5\pi}}(Z-2) R_0^2 \beta_2 q_0, \tag{VI-2}$$

where Z is the atomic number of the parent nucleus and q_0 is a dimensionless quantity which depends on the charge distribution of the nucleus. The value of β_2 can be estimated from the formula (VI-2) by assuming the nucleus to have ellipsoidal shape and uniform charge density, in which case q_0 is equal to 1.

The differential equation (V-24) can be written

$$\Delta \psi = K^2 \psi, \tag{VI-3}$$

where***

$$K = K(\boldsymbol{r}') = k \sqrt{\frac{V(\boldsymbol{r}') - E_{K_f}}{E_{K_f}}}, \tag{VI-4}$$

if k here denotes the magnitude of the wave vector corresponding to the energy E_{K_f}. According to (V-4), (V-10), (V-12), (VI-4), and (A-4) we get

$$K(\boldsymbol{r}') = k \sqrt{\frac{\varkappa}{kr} - 1} \left\{ 1 + \frac{4\pi}{Z-2} \sum_{\lambda > 0} \frac{1}{2\lambda+1} \left(\frac{k}{\varkappa}\right)^{\lambda} \frac{Q'_\lambda}{\varepsilon} \left(\frac{kr}{\varkappa}\right)^{-\lambda-1} \left(\frac{\varkappa}{kr} - 1\right)^{-1} Y_{\lambda,0}(\vartheta') \right\}^{\frac{1}{2}}. \tag{VI-5}$$

It is convenient to write

$$K(\boldsymbol{r}') = K_0(r) + \Delta K(r, \vartheta'), \tag{VI-6}$$

where

$$K_0(r) = k \sqrt{\frac{\varkappa}{kr} - 1} \tag{VI-7}$$

and $\Delta K(r, \vartheta')$ is the purely anisotropic part of $K(\boldsymbol{r}')$. From (VI-5), (VI-6), and (VI-7) we easily find the approximate formula

$$\Delta K(r, \vartheta') = k \frac{2\pi}{Z-2} \sum_{\lambda > 0} \frac{1}{2\lambda+1} \left(\frac{k}{\varkappa}\right)^{\lambda} \frac{Q'_\lambda}{\varepsilon} \left(\frac{kr}{\varkappa}\right)^{-\lambda-1} \left(\frac{\varkappa}{kr} - 1\right)^{-\frac{1}{2}} Y_{\lambda,0}(\vartheta'). \tag{VI-8}$$

* Our deformation parameter β_2 corresponds to the quantity $\beta \cos \gamma$ in the papers by Bohr and Mottelson[14, 16] where β is positive and γ is equal to 0 for an axially symmetric prolate nucleus and equal to π for an axially symmetric oblate nucleus.

** See Eq. (V. 7) on p. 56 in ref. 16 or Eqs. (V-13) and (V-16) in the present paper.

*** See footnote ** on p. 16.

The differential equation (VI-3), which we use in the region between the nuclear surface and the surface of the sphere of radius R_1, is to be solved under the boundary conditions that, on the nuclear surface, the wave function ψ shall be equal to a given function $\psi_0(\vartheta', \varphi')$ depending on Z, A, K_i, and K_f, but not on I_i and I_f, and that ψ shall decrease very rapidly as one moves outwards in the potential barrier. An approximate solution of this problem can be obtained by arguing as follows. The anisotropic part of the electrostatic field acts essentially only at points comparatively near to the nucleus, whereas the centrifugal barrier is of importance over larger distances. In first approximation, we may therefore neglect the centrifugal barrier at those small distances where the anisotropy is of importance. The fact that this approximation is not a very good one is not particularly dangerous, for it turns out that the nuclear shape is more important for the alpha decay than is the anisotropy of the electrostatic field. From these arguments it follows that, if we want to calculate the alpha wave function at a point at a distance $r\ (\leq R_1)$ from the center of the nucleus, we can in first approximation replace the actual nuclear surface by the sphere of radius R_0 and the actual anisotropic potential barrier by the corresponding spherically symmetric potential barrier, if we at the same time replace the alpha wave function $\psi_0(\vartheta', \varphi')$ on the actual nuclear surface by the alpha wave function given by (B-34) on the sphere of radius R_0. In the bracket of (B-34) the first term is due to the non-spherical nuclear shape, whereas the second term (which is less important) is due to the anisotropy of the potential barrier. For the alpha wave function ψ at a point* (r, ϑ, φ) such that $r \leq R_1$ we now easily realize that the formula (B-35) applies. This formula, which we have here deduced in a more or less intuitive way, is obtained, in Appendix B, as an approximate result of a method which, in principle, is more general and less approximate.

From (V-25), (VI-1), (VI-6), (VI-7), (VI-8), and (B-35) we get

$$
\begin{aligned}
& g_{K_f, l, \Omega}(r) \\
&= R_0 \frac{G_l(E_{K_f}, r)}{G_l(E_{K_f}, R_0)} \int_0^{\pi}\int_0^{2\pi} \psi_0(\vartheta, \varphi) \exp\left\{\int_{R_0}^{R(\vartheta)} K(r, \vartheta)\, dr - \int_{R_0}^{r} \Delta K(r, \vartheta)\, dr\right\} Y^*_{l,\Omega}(\vartheta, \varphi) \sin\vartheta\, d\vartheta\, d\varphi \\
&= R_0 \frac{G_l(E_{K_f}, r)}{G_l(E_{K_f}, R_0)} \int_0^{\pi}\int_0^{2\pi} \psi_0(\vartheta, \varphi) \exp\left\{\varkappa \sum_{\lambda>0} \left(\beta_\lambda \sqrt{\frac{k R_0}{\varkappa}\left(1 - \frac{k R_0}{\varkappa}\right)}\right.\right. \\
&\quad \left.\left. - \frac{2\pi}{Z-2} \frac{1}{2\lambda+1} \left(\frac{k}{\varkappa}\right)^{\lambda} \frac{Q'_\lambda}{\varepsilon} \int_{\frac{kR_0}{\varkappa}}^{\frac{kr}{\varkappa}} \frac{dx}{x^\lambda \sqrt{x(1-x)}}\right) Y_{\lambda,0}(\vartheta)\right\} Y^*_{l,\Omega}(\vartheta, \varphi) \sin\vartheta\, d\vartheta\, d\varphi
\end{aligned}
$$

* Here, the unprimed polar angles ϑ and φ are assumed to be referred to a coordinate system with axes having directions which are fixed with respect to the nucleus.

for $r \leqslant R_1$. In the exponent of this expression, the term corresponding to $\lambda = 2$ can be simplified by means of (V-13) and (VI-2), and for $r = R_1$ the upper integration limit $kr/\varkappa$ may approximately be replaced by 1. Therefore we get

$$g_{K_f, l, \Omega}(R_1) = R_0 \frac{G_l(E_{K_f}, R_1)}{G_l(E_{K_f}, R_0)} \int_0^{\pi}\int_0^{2\pi} \psi_1(\vartheta, \varphi) \exp\{B P_2(\cos\vartheta)\} Y^*_{l,\Omega}(\vartheta, \varphi) \sin\vartheta\, d\vartheta\, d\varphi,$$

where

$$B = \varkappa \beta_2 \sqrt{\frac{5}{4\pi}} \sqrt{\frac{kR_0}{\varkappa}\left(1 - \frac{kR_0}{\varkappa}\right)} \left(1 - \frac{1}{5} q_0 - \frac{2}{5} q_0 \frac{kR_0}{\varkappa}\right) \tag{VI-9}$$

and

$$\left.\begin{aligned} \psi_1(\vartheta, \varphi) = \psi_0(\vartheta, \varphi) \exp\Bigg\{ \varkappa \sum_{\substack{\lambda>0 \\ \lambda \neq 2}} \Bigg(\beta_\lambda \sqrt{\frac{kR_0}{\varkappa}\left(1 - \frac{kR_0}{\varkappa}\right)} \\ - \frac{2\pi}{Z-2} \frac{1}{2\lambda+1} \left(\frac{k}{\varkappa}\right)^{\lambda} \frac{Q'_\lambda}{\varepsilon} \int_{\frac{kR_0}{\varkappa}}^{1} \frac{dx}{x^\lambda \sqrt{x(1-x)}} \Bigg) Y_{\lambda,0}(\vartheta) \Bigg\}, \end{aligned}\right\} \tag{VI-10}$$

the second term in the parenthesis ( ) in the exponent of (VI-10) being less than 20 % of the first term in this parenthesis if $\lambda > 2$. The function $\psi_1(\vartheta, \varphi)$ is now assumed to be resolved in terms of spherical harmonics as follows

$$\psi_1(\vartheta, \varphi) = \sum_{l'} \sum_{\Omega'} a_{l', \Omega'} Y_{l', \Omega'}(\vartheta, \varphi), \tag{VI-11}$$

where $a_{l', \Omega'}$ are coefficients which depend on $E_{K_f}, Z, A, \beta_3, \beta_4, \ldots, Q'_1, Q'_3, Q'_4, \ldots$ and on the parameters appearing in $\psi_0(\vartheta, \varphi)$. Since the component of the total angular momentum along the nuclear axis is conserved during formation of the alpha particle, the coefficients $a_{l, \Omega}$ are different from zero only if $\Omega = \pm K_i - K_f$ and $|\Omega| \leqslant l$. Using (III-10), (VI-11), and the formula for $g_{K_f, l, \Omega}(R_1)$ on this page, we get

$$f_{K_f, l, I_f}(R_1) = R_0 \frac{G_l(E_{K_f}, R_1)}{G_l(E_{K_f}, R_0)} \sum_{\Omega} (-1)^{I_f - I_i + \Omega} (I_i, l; K_f + \Omega, -\Omega \,|\, I_i, l; I_f, K_f) \sum_{l'} k^{\Omega}_{l, l'}(B)\, a_{l', \Omega},$$

where the quantities $k^{\Omega}_{l, l'}(B)$ are the elements of a symmetric matrix defined by

$$k^{\Omega}_{l, l'}(B) = \int_0^{\pi} \Theta_{l, \Omega}(\vartheta) \exp\{B P_2(\cos\vartheta)\} \Theta_{l', \Omega}(\vartheta) \sin\vartheta\, d\vartheta,$$

the normalized ϑ-depending factor in the spherical harmonic $Y_{l, \Omega}(\vartheta, \varphi)$ being denoted by $\Theta_{l, \Omega}(\vartheta)$. According to Fig. 6 (in Appendix A), the function $G_l(E, R_1)/G_l(E, R_0)$ does not change as rapidly with E in the interior region of the potential barrier, as one might expect from the strong E-dependence of $G_l(E, r)$. In the functions G_l appearing in the above formula for $f_{K_f, l, I_f}(R_1)$, we may therefore approximately replace* E_{K_f} by E_{I_f} which is an abbreviation for E_{K_f, I_f}. Using the formula for $f_{K_f, l, I_f}(R_1)$ thus obtained and noting that for $r \geqslant R_1$ the function $f_{K_f, l, I_f}(r)$ is approximately equal to $G_l(E_{I_f}, r)$, except for a factor which does not depend on r, we get the formula

$$\left.\begin{aligned} & f_{K_f, l, I_f}(r) \\ & = R_0 \frac{G_l(E_{I_f}, r)}{G_l(E_{I_f}, R_0)} \sum_{\Omega} (-1)^{I_f - I_i + \Omega} (I_i, l; K_f + \Omega, -\Omega \,|\, I_i, l; I_f, K_f) \sum_{l'} k^{\Omega}_{l, l'}(B)\, a_{l', \Omega} \end{aligned}\right\} \quad \text{(VI-12)}$$

which is valid for $r \geqslant R_1$.

From (IV-1), (VI-12), (A-15a), and (A-11b), we find that the alpha transition probability per unit time for the transition from the state (K_i, I_i) of the parent nucleus to the state (K_f, I_f) of the daughter nucleus is

$$\left.\begin{aligned} P_{K_f, I_f} = \sum_l P_{K_f, l, I_f} = v_{I_f} \left(\frac{R_0}{G_0(E_{I_f}, R_0)} \right)^2 \sum_l \exp\left\{ - \frac{2\, l(l+1)}{\varkappa} \left(\sqrt{\frac{\varkappa}{k R_0}} - 1 \right) \right\} \\ \times \left| \sum_{\Omega} (-1)^{\Omega} (I_i, l; K_f + \Omega, -\Omega \,|\, I_i, l; I_f, K_f) \sum_{l'} k^{\Omega}_{l, l'}(B)\, a_{l', \Omega} \right|^2. \end{aligned}\right\} \quad \text{(VI-13)}$$

According to (IV-2), (VI-12), (A-15a), and (A-11b), the probability per unit time and unit solid angle (in a center-of-mass coordinate system with axes having directions fixed in space) that an alpha particle leaks out in the direction (ϑ, φ) leaving the daughter nucleus in the state (K_f, I_f) is obtained by averaging the following expression with respect to the initial distribution of M_i-values:

$$\left.\begin{aligned} & \frac{1}{2\pi} v_{I_f} \left(\frac{R_0}{G_0(E_{I_f}, R_0)} \right)^2 \sum_{M_f} \left| \sum_l \exp\left\{ - \frac{l(l+1)}{\varkappa} \left(\sqrt{\frac{\varkappa}{k R_0}} - 1 + i \right) \right\} \right. \\ & \qquad \times (I_f, l; M_f, M_i - M_f \,|\, I_f, l; I_i, M_i)\, \Theta_{l, M_i - M_f}(\vartheta) \\ & \left. \times \sum_{\Omega} (-1)^{\Omega} (I_i, l; K_f + \Omega, -\Omega \,|\, I_i, l; I_f, K_f) \sum_{l'} k^{\Omega}_{l, l'}(B)\, a_{l', \Omega} \right|^2. \end{aligned}\right\} \quad \text{(VI-14)}$$

For $K_i = 0$ the sum over Ω in (VI-12), (VI-13), and (VI-14) contains only one term (corresponding to $\Omega = -K_f$). For $K_i \neq 0$ the same sum may contain two terms (corresponding to $\Omega = K_i - K_f$ and $\Omega = -K_i - K_f$) if $l \geqslant K_i + K_f$, one term (corresponding to $\Omega = K_i - K_f$) if $|K_i - K_f| \leqslant l < K_i + K_f$, whereas it is zero if $l < |K_i - K_f|$. Furthermore, l is restricted either to even values only or to odd values only, depending on whether the parent nucleus and the band K_f of the daughter nucleus have the same or opposite parity.

A detailed theoretical estimate of the accuracy of the approximations involved in the derivation of (VI-12), from which (VI-13) and (VI-14) are obtained, is dif-

* The justification for this replacement may be somewhat improved by keeping ΔE_{K_f} in (V-24) and choosing it such that $E_{K_f} - \Delta E_{K_f}$ is some suitable average value of the E_{I_f}-values considered. See the footnote * on p. 16.

ficult, but a very crude estimate indicates that, for the typical values $Z \approx 90$, $Q_0 \approx 10 \cdot 10^{-24}\,\mathrm{cm}^2$, $E_{K_f} \approx 5$ MeV, and $\left| E_{I_f} - E_{K_f} \right| \lesssim 100$ keV, the error involved is less than about 50 $^0/_0$, and it seems quite probable that it is appreciably smaller. An essential part of this error is due to the approximation that the penetration of the alpha particle through the interior region of the potential barrier was assumed to take place in the same way as if the nucleus were fixed in space. Another essential part of the error is due to the neglect of the anisotropy of the potential barrier in the region outside of $r = R_1$. The accuracy of (VI-12) is expected to decrease as $I_f - K_f$ increases, since, firstly, the approximation which is involved in the replacement of (V-20) by (V-24) in the region between the nuclear surface and the sphere of radius R_1 as well as the replacement of $G_l\,(E_{K_f},\,R_1)$ by $G_l\,(E_{I_f},\,R_1)$ becomes less accurate as the energy differences between the different alpha groups considered increase; secondly, in the region $r > R_1$, the neglected coupling between the different functions $f_{K_f,\,l,\,I_f}(r)$ should be comparatively more important for the smaller than for the larger of these functions.

In a few special cases, the coupled differential equations which describe the alpha decay of even-even nuclei of ellipsoidal shape have been solved numerically by RASMUSSEN and SEGALL[45]. These authors treat the rotational energy term by an approximation method and take into account only the partial waves corresponding to alpha groups of angular momenta $l \leq 4$. Their results may therefore possibly contain a minor uncertainty in the intensity ratio for the alpha groups α_2 and α_0 and a somewhat larger uncertainty in the intensity ratio for the alpha groups α_4 and α_0. In two cases corresponding to β_2-values of 0.30 (their $\xi_0 = 1.51$) and 0.36 (their $\xi_0 = 1.41$), the alpha wave function was assumed to be constant on the nuclear surface. For these two cases the intensity ratios calculated by RASMUSSEN and SEGALL (cf. their table I) can be compared with the corresponding intensity ratios calculated according to the intensity formula in the present paper, and it is found that the agreement is good to within 50 $^0/_0$. For the case corresponding to the smaller one of the above mentioned β_2-values (i. e., to $\beta_2 = 0.30$), the agreement for the ratio of the intensities of the alpha groups α_2 and α_0 is even somewhat better than 10 $^0/_0$.

VII. Analysis of the Experimental Data.*

The available experimental data on alpha decay of parent nuclei with known alpha branching** and with $Z > 82$ (except for the data on long range alpha particles) are collected in Table 1. Most data are taken from the table of isotopes by HOLLANDER, PERLMAN, and SEABORG[35], which in the following is referred to as HPS. In those cases, where some or all data come from other publications, the last column of Table 1 gives

* Note added in proof: Further data are given in the forthcoming review article on alpha decay by PERLMAN and RASMUSSEN[43a] (see their table I). When the analysis in the present paper was made, their data had not yet been available.

** As regards alpha decay of nuclei with unknown alpha branching, see, e. g., the recent work on the light Po-isotopes by ROSENBLUM and TYRÉN[48].

the appropriate references in terms of figures referring to the list at the end of the present paper. HPS is usually not quoted, even if data from this table are used to complete data taken from other publications. Thus, almost all the half-lives are cited according to HPS. For those parent nuclei which are quoted without any reference, all the data have been obtained from HPS. When some remark is required, the last column of Table 1 contains a letter which refers to the list of remarks on p. 33.

In the first column of Table 1, the symbols for the alpha radioactive parent nuclei with $Z > 82$ are listed. The third column gives the experimentally measured half-life T_{tot} of each parent nucleus. This half-life may result from different kinds of radioactive branching. The second column gives the percentage of radioactivity that is alpha decay. For some parent nuclei, column two shows only lower boundaries for the percentage of alpha branching. If this lower boundary is rather close to 100, we use the mean value of 100 and the boundary in the calculations that are required to obtain the values in some of the following columns. If, on the other hand, the lower boundary is close to 0, we use the value of the boundary itself. (Example: If the second column gives $\geqslant 90$, we have chosen the value 95 (= mean value of 90 and 100). If the second column gives $\geqslant 10$, we have chosen the value 10). When the second column is empty, it is understood that the alpha branching is given as 100 %. In the fourth column, the energies E_α (expressed in MeV) of the alpha particles (except for the long range alpha particles) are listed. It should be mentioned that the listed energies are the alpha particle energies and not the transition energies*. This choice was made only in order to simplify the numerical work. For most parent nuclei, the alpha particle energies are found in HPS or in the references quoted in the last column of Table 1; in some cases, however, where these references give, instead, the energies of the gamma rays following the alpha decay, the alpha particle energies E_α have been calculated from the energy $E_\alpha^{\max}$ of the most energetic alpha particles and the excitation energies E_{exc} of the daughter nucleus. This is easily done by means of the formula

$$E_\alpha = E_\alpha^{\max} - E_{\text{exc}} + \frac{m}{M} E_{\text{exc}},$$

where M is the mass of the daughter nucleus, and m is the reduced mass of the alpha particle and the daughter nucleus. The last term on the right-hand side of this formula is sometimes unimportant, and in such cases we have often neglected it entirely.

For every parent nucleus, the number of alpha particles in a particular group relative to the total number of alpha particles is called the relative intensity or the abundance of the alpha group. The experimentally determined values of these relative intensities (expressed in per cent) are listed in the fifth column. For those parent nuclei for which only one alpha group has been observed this column is empty. For every parent nucleus the sum of the percentages in column five should be 100; due to experimental uncertainties this is only approximately true for the values listed in Table 1.

* As will be remembered, the transition energy of an alpha group has been defined as the sum of the energy of the alpha particle and the recoil energy of the daughter nucleus.

The partial half-life T (expressed in seconds) of an alpha group is the product of three factors. The first factor is the experimentally measured half-life T_{tot} in column three (expressed in seconds), the second one is 100 divided by the alpha branching percentage in column two, and the third one is 100 divided by the relative intensity percentage in column five. Column six gives the values of $\log_{10} T$, T being expressed in seconds*. These values are listed with two decimals, although in general the accuracy of the experiments is not as good. Negative logarithms are recognized by an index n, so that, e. g., $\log_{10} T = 8.28_n$ means $\log_{10} T = 8.28 - 10$.

If one plots $\log_{10} T$ against $E_\alpha^{-\frac{1}{2}}$ for the ground state alpha transitions of the even-even nuclei, one finds that the points corresponding to a fixed value of the atomic number Z of the parent nuclei lie almost on a straight line, except for those parent nuclei which are too near to the closed shell with $Z = 82$ protons and $N = 126$ neutrons (i. e., $A = Z + N = 208$). This plot is shown in Fig. 1, where the symbol referring to each point denotes the corresponding parent nucleus. For those even-even parent nuclei (of Po and Em) which have mass numbers $A < 214$, appreciable deviations from the straight lines occur, and therefore these nuclei are not included here. Considered individually, some of the straight lines may not be very accurately defined, but if one considers them collectively and requires that the distances between any two neighbouring lines and also the slopes of the lines shall vary regularly with Z, all the straight lines are well enough defined for our purpose. In drawing the lines one should, of course, take into account the further requirement that those points which are most accurately known from experiments must be especially well approximated by the straight lines. The straight lines are drawn graphically in such a way that they fulfill as well as possible all these requirements. As is easily seen from the figure, the distance between any two successive straight lines decreases slightly and regularly as Z increases. The accuracy of the experimental data available does not allow us to draw any regular curves that are better than straight lines. The reason why a few of the even-even parent nuclei (especially Pu^{232}) show more essential deviations from the straight lines may be associated with uncertainties in the experimental data for these nuclei, and it would therefore be important to re-examine these alpha decays.

The smooth function of E_α and Z which gives the partial half-lives for the ground state alpha transitions of even-even nuclei will be denoted by T_0. The analytical formula

$$\left.\begin{aligned} \log_{10} T_0 = \left\{139.8 + 1.83\,(Z-90) + 0.012\,(Z-90)^2\right\} \frac{1}{\sqrt{E_\alpha}} \\ - \left\{52.3 + 0.30\,(Z-90) + 0.001\,(Z-90)^2\right\}, \end{aligned}\right\} \qquad \text{(VII-1)}$$

where T_0 is expressed in seconds and E_α in MeV, gives a good approximation of the straight lines in Fig. 1. The experimental data on the ground state alpha transitions of the even-even nuclei are, in fact, equally well represented by the formula (VII-1)

* The values in columns six to nine have been obtained without regard to the signs $\geqslant$, $>$, and $\sim$ which appear for some data in columns two to five.

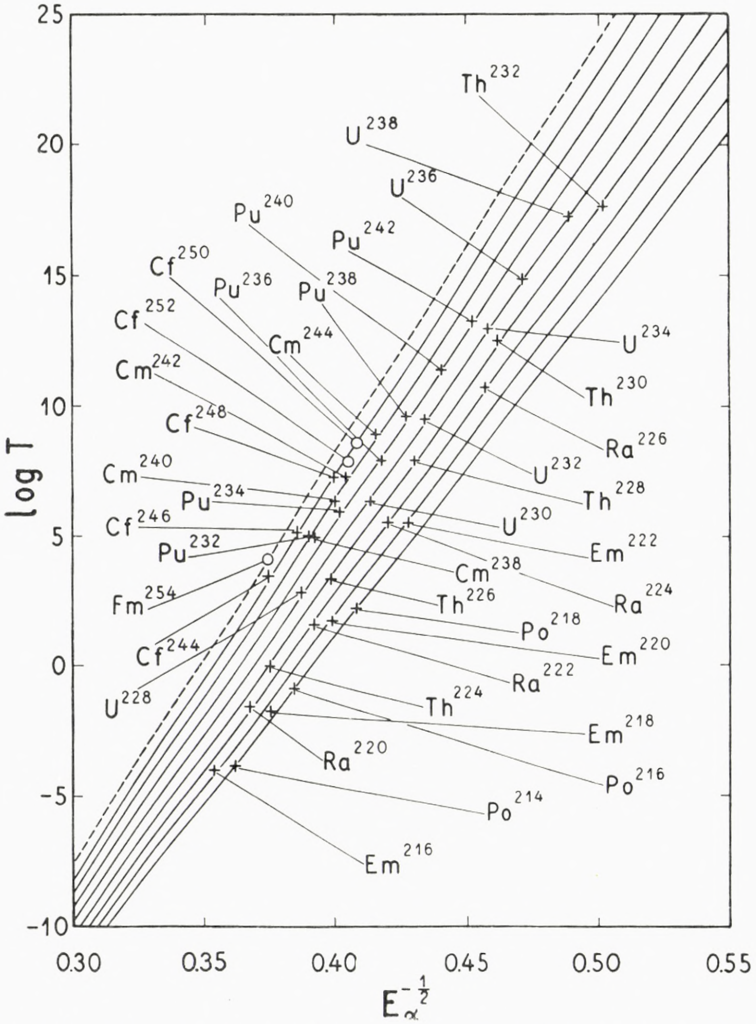

Fig. 1. The 10-logarithms of the partial half-lives T (expressed in seconds) for the ground state alpha transitions of the even-even nuclei have been plotted against $E_\alpha^{-\frac{1}{2}}$, where E_α is the corresponding alpha particle energy expressed in MeV. The chemical symbols referring to the experimental points denote the parent nuclei. The straight lines have been drawn graphically such that the points corresponding to a fixed Z-value shall be approximated as well as possible by a straight line and that the sequence of straight lines shall be spaced as regularly as possible. For the parent nuclei Em^{218}, Ra^{222}, Th^{226}, U^{230}, U^{232}, Cf^{244}, and Cf^{246} new data have appeared after this figure was drawn. The changes in the corresponding experimental points which are implied by these new data are small, and therefore we have not re-drawn the figure. For the parent nuclei Cf^{250}, Cf^{252}, and Fm^{254}, no experimental data were available when this figure was drawn. The points corresponding to these three parent nuclei have later been included in the figure, where they are recognized as small circles. Hence the system of straight lines corresponding to $Z = 84, 86, \ldots, 98$ as well as the dotted line corresponding to $Z = 100$ (which has been obtained by extrapolating the regular sequence of lines corresponding to $Z = 84, 86, \ldots, 98$) was drawn before any data on the three parent nuclei Cf^{250}, Cf^{252}, and Fm^{254} were available.

The signs $\geq$ and $>$ which appear in column two of Table 1 indicate that, in the above figure, the points corresponding to the parent nuclei Pu^{232} and Cm^{238} may be displaced somewhat downwards (cf. footnote on p. 23).

After this figure had been prepared for publication, there appeared data on the ground state alpha transitions of the parent nuclei Cm^{246} (cf. refs. 19a and 43a), Cm^{248} (cf. ref. 19a), Fm^{250} (cf. ref. 43a), and Fm^{252} (cf. Table 1 and ref. 43a).

as by the straight lines in Fig. 1. The absolute value of the difference between $\log_{10} T_0$ according to the formula (VII-1) and $\log_{10} T_0$ according to the full-drawn straight lines in the figure is less than 0.1 if $E_\alpha^{-\frac{1}{2}}$ lies between 0.35 and 0.50, i. e., if E_α lies between 4.0 and 8.2 MeV.

The formula

$$\left.\begin{aligned}\log_{10} T_0 = \left\{140 + 1.8\,(Z-90) + 0.01\,(Z-90)^2\right\}\frac{1}{\sqrt{E_\alpha}} \\ - \left\{52.3 + 0.3\,(Z-90)\right\},\end{aligned}\right\} \qquad \text{(VII-2)}$$

which is an approximation of (VII-1), is also useful. For the parent nuclei of the isotopes of Th, U, Pu, Cm, Cf, and Fm in Fig. 1, the experimental data on the ground state alpha transitions are about equally well represented by (VII-2) as by the straight lines in Fig. 1, and for the parent nuclei of the isotopes of Po, Em, and Ra in Fig. 1 the experimental data are only slightly better represented by the straight lines in Fig. 1 than by (VII-2). The absolute value of the difference between $\log_{10} T_0$ according to (VII-2) and $\log_{10} T_0$ according to the straight lines in Fig. 1 is less than 0.2 if $E_\alpha^{-\frac{1}{2}}$ lies between 0.35 and 0.50, i. e., if E_α lies between 4.0 and 8.2 MeV.

A priori the smoothed out function T_0 is, of course, defined only for even Z-values, but it is very useful to define this function also for odd Z-values. Fig. 2 shows the straight lines obtained for even Z-values in Fig. 1 together with the straight lines for odd Z-values found by graphic interpolation. The system of straight lines in Fig. 2 defines T_0 graphically as a smooth function of E_α and Z both for even and odd values of Z. Column seven of Table 1 gives the values of $\log_{10} T_0$ (with T_0 expressed in seconds) which have been obtained graphically from Fig. 2 by putting Z equal to the atomic number of the parent nucleus and E_α equal to the alpha particle energy of the alpha group considered. These values of $\log_{10} T_0$ are listed with two decimals, the second decimal being estimated as either 0 or 5. The intrinsic transition probability for an alpha group is given by the quotient T_0/T which we shall call the F-value of the alpha group. From columns six and seven of Table 1 we have calculated the values of $\log_{10}(T_0/T)$ to one decimal, and from these values we have obtained the F-values which are listed in column eight. The absolute F-values are approximate to the same degree as the function T_0, but the relative F-values for the alpha groups of any given parent nucleus are much more accurate. For favoured alpha groups such relative F-values are very useful. They will be designated as the reduced transition probabilities and will be denoted by c and defined as the quotient of the F-value for the alpha group considered and the F-value for the special alpha transition of the same nucleus for which $\Delta I = 0$. The F-values appearing in this definition are best calculated by means of (VII-1) or (VII-2), for then one avoids errors due to the difficulty of reading the graph in Fig. 2 accurately. The reduced transition probabilities c, obtained by means of (VII-1), for the alpha groups of even-even nuclei and the favoured alpha groups of odd-A nuclei are listed in column nine. As regards

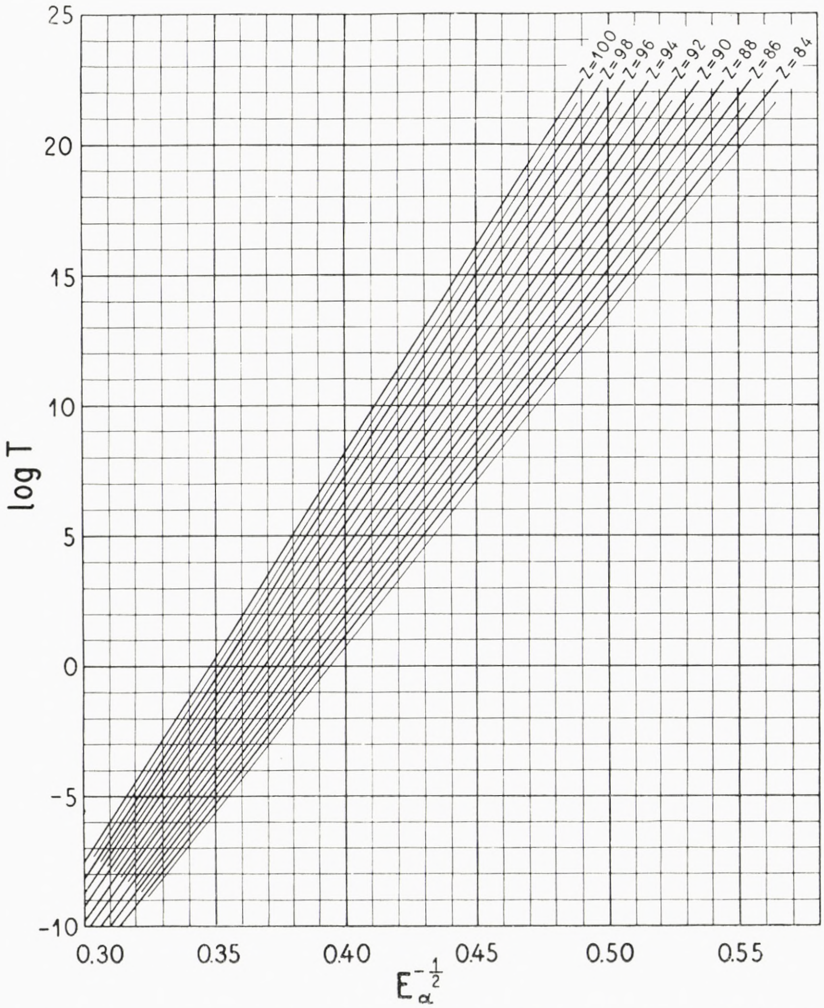

Fig. 2. The straight lines corresponding to even values of the atomic number Z of the parent nucleus are the same as those in Fig. 1; the straight lines corresponding to odd values of Z have been obtained by graphic interpolation between the lines corresponding to even Z-values.

the numerical values of c compared with those of F it is clear that, on the average, $F \approx c$ for the alpha groups of even-even nuclei and, as shown in Table 1, $F \approx 0.5c$ for the favoured alpha groups of odd-A nuclei.

Note added in proof: In the forthcoming review article on alpha decay by PERLMAN and RASMUSSEN[43a], an analysis of the empirical alpha decay data is also presented. It is essentially similar to ours, although it is partly based on quite recent data which had not been available to us. Despite some minor differences between the two analyses, the results are in general in satisfactory agreement.

TABLE 1. Analysis of the experimental data on alpha decay for parent nuclei with $Z > 82$ (except for the data on long range alpha particles).

Parent nucleus	$^0/_0\ \alpha$	T_{tot}	E_α	Rel. α-int. ($^0/_0$)	$\log_{10} T$	$\log_{10} T_0$	F	c	References and remarks
$_{83}Bi^{<198}$		1.7 m	6.2		2.01	0.95	0.08		
Bi^{198}	$5\cdot10^{-2}$	7 m	5.83		5.92	2.60	0.0005		
Bi^{199}	10^{-2}	$\sim$ 25 m	5.47		7.18	4.30	0.001		
Bi^{201}	$3\cdot10^{-3}$	62 m	5.15		8.09	5.95	0.008		
Bi^{203}	$\sim 10^{-5}$	12 h	4.85		11.63	7.65	0.0001		
Bi^{209}		$3\cdot10^{17}$ y	$\sim$ 3.15		24.98	21.55	0.0004		
Bi^{211}	99.68	2.16 m	6.618	84	2.19	9.20_n	0.001		
			6.272	16	2.91	0.65	0.005		
Bi^{212}	33.7	60.5 m	6.086	27.2	4.60	1.50	0.0008		
			6.047	69.9	4.19	1.60	0.003		
			5.765	1.7	5.80	2.85	0.001		
			5.622	0.15	6.86	3.55	0.0005		
			5.603	1.1	5.99	3.65	0.005		
			5.481	0.016	7.83	4.30	0.0003		
Bi^{213}	2	46.5 m	5.86		5.14	2.45	0.002		
Bi^{214}	0.04	19.7 m	5.505	45	6.82	4.15	0.002		
			5.444	55	6.73	4.40	0.005		
$_{84}Po^{204}$	$\sim$ 1	3.8 h	5.37		6.13	5.30	0.2		
Po^{206}	$\sim$ 10	9 d	5.218		6.89	6.10	0.2		
Po^{207}	$\sim 10^{-2}$	5.7 h	5.10		8.31	6.80	0.03		
Po^{208}		2.93 y	5.109		7.97	6.70	0.05		
Po^{209}	≥ 90	$\sim$ 100 y	4.877		9.52	8.05	0.03		
Po^{210}		139 d	5.301		7.08	5.60	0.03		
Po^{211}		0.52 s	7.434	99	9.72_n	6.85_n	0.001		
			6.88	0.50	2.02	8.80_n	0.0006		
			6.56	0.53	1.99	0.00	0.01		
Po^{212}		$3.0\cdot10^{-7}$ s	8.776		3.48_n	3.10_n	0.4		
Po^{213}		$4.2\cdot10^{-6}$ s	8.336		4.62_n	4.30_n	0.5		
Po^{214}		$1.5\cdot10^{-4}$ s	7.680		6.18_n	6.15_n	1		
Po^{215}	≥ 99	$1.83\cdot10^{-3}$ s	7.365		7.27_n	7.10_n	0.6		
Po^{216}	≥ 99	0.158 s	6.774		9.20_n	9.20_n	1		
Po^{218}	≥ 99	3.05 m	5.998		2.26	2.25	1		
$_{85}At^{207}$	$\sim$ 10	2.0 h	5.75		4.86	3.80	0.08		
At^{208}	0.5	1.7 h	5.65		6.09	4.25	0.02		
At^{209}	$\sim$ 5	5.5 h	5.65		5.60	4.25	0.05		
At^{210}	0.17	8.3 h	5.519	32	7.74	4.90	0.002		
			5.437	31	7.75	5.30	0.004		
			5.355	37	7.68	5.80	0.01		
At^{211}	40.9	7.4 h	5.862		4.81	3.30	0.03		
At^{214}		$\sim 2\cdot10^{-6}$ s	8.78		4.30_n	3.35_n	0.1		
At^{215}		$\sim 10^{-4}$ s	8.00		6.00_n	5.50_n	0.3		
At^{216}		$\sim 3\cdot10^{-4}$ s	7.79		6.48_n	6.10_n	0.4		
At^{217}		0.019 s	7.01		8.28_n	8.65_n	2.5		
At^{219}	$\sim$ 97	0.9 m	6.27		1.75	1.50	0.6		

TABLE 1 (continued).

Parent nucleus	$^0/_0\,\alpha$	T_{tot}	E_α	Rel. α-int. ($^0/_0$)	$\log_{10} T$	$\log_{10} T_0$	F	c	References and remarks
$_{86}Em^{208}$	~ 20	23 m	6.138		3.84	2.45	0.04		
Em^{209}	~ 20	30 m	6.02		3.95	3.00	0.1		
Em^{210}	> 95	2.5 h	6.036		3.96	2.90	0.08		
Em^{211}	25	16 h	5.847	33	5.84	3.80	0.01		
			5.778	67	5.54	4.10	0.04		
Em^{212}		23 m	6.262		3.14	1.95	0.06		
Em^{215}		$\sim 10^{-6}$ s	8.6		4.00_n	4.10_n	1.3		
Em^{216}		$\sim 10^{-4}$ s	8.01		6.00_n	5.80_n	0.6		
Em^{217}		$\sim 10^{-3}$ s	7.74		7.00_n	6.60_n	0.4		
Em^{218}		0.019 s	7.13	99.8	8.28_n	8.60_n	2.0	1	11, 42
			6.53	0.2	0.98	0.85	0.8	0.34	
Em^{219}		3.92 s	6.824	69	0.75	9.70_n	0.1		
			6.559	15	1.42	0.75	0.2		
			6.434	12	1.51	1.30	0.6		
			6.214	4	1.99	2.15	1.6		
Em^{220}		54.5 s	6.282		1.74	1.85	1.3		
Em^{221}	20	25 m	6.0		3.87	3.10	0.2		40
Em^{222}		3.825 d	5.486		5.52	5.55	1		
$_{87}Fr^{212}$	44	19.3 m	6.409	37	3.85	1.75	0.008		
			6.387	39	3.83	1.80	0.01		
			6.339	24	4.04	2.00	0.01		
Fr^{218}		$\sim 5\cdot 10^{-3}$ s	7.85		7.70_n	6.60_n	0.08		
Fr^{219}		0.02 s	7.30		8.30_n	8.35_n	1.3		
Fr^{220}		27.5 s	6.69		1.44	0.60	0.2		
Fr^{221}		4.9 m	6.30	~ 75	2.59	2.20	0.4		
			6.05	~ 25	3.07	3.30	1.6		
$_{88}Ra^{213}$		2.7 m	6.90		2.21	0.20	0.01		
Ra^{219}		$\sim 10^{-3}$ s	8.0		7.00_n	6.50_n	0.3		
Ra^{220}		$\sim 3\cdot 10^{-2}$ s	7.43		8.48_n	8.30_n	0.6		
Ra^{221}		30 s	6.71		1.48	0.90	0.3		
Ra^{222}		38 s	6.55	95	1.60	1.55	1	1	11, 42
			6.22	5	2.88	2.95	1.3	1.35	
Ra^{223}		11.2 d	5.730	9	7.03	5.30	0.02		a
			5.704	53	6.26	5.40	0.1		
			5.596	24	6.60	5.90	0.2		
			5.528	9	7.03	6.30	0.2		
			5.487	2	7.68	6.50	0.06		
			5.419	3	7.51	6.85	0.2		
Ra^{224}		3.64 d	5.681	95	5.52	5.50	1	1	4
			5.445	5	6.80	6.70	0.8	0.87	
Ra^{226}		1622 y	4.777	94.3	10.73	10.65	0.8	1	
			4.592	5.7	11.95	11.85	0.8	1.05	

Table 1 (continued).

Parent nucleus	$^0/_0\,\alpha$	T_{tot}		E_α	Rel. α-int. ($^0/_0$)	$\log_{10} T$	$\log_{10} T_0$	F	c	References and remarks
$_{89}Ac^{222}$		5.5	s	6.96		0.74	0.35	0.4		
Ac^{223}	99	2.2	m	6.64		2.12	1.60	0.3		
Ac^{224}	~ 10	2.9	h	6.17		5.02	3.55	0.03		
Ac^{225}		10.0	d	5.80		5.94	5.35	0.3		
Ac^{227}	1.2	22	y	4.942		10.76	10.05	0.2		
$_{90}Th^{223}$		~ 0.1	s	7.55		9.00_n	8.65_n	0.5		
Th^{224}		~ 1	s	7.13		0.00	0.10	1.3		
Th^{225}	~ 90	8.0	m	6.57		2.73	2.30	0.4		
Th^{226}		30.9	m	6.33	79	3.37	3.30	0.8	1	11, 42, b
				6.22	19	3.99	3.80	0.6	0.74	
				6.095*	1.7	5.03	4.40	0.3	0.25	
				6.03	0.6	5.49	4.65	0.2	0.17	
Th^{227}		18.6	d	6.030	19	6.93	4.70	0.006		c
				6.001	5	7.51	4.80	0.002		
				5.972	21	6.88	4.95	0.01		
				5.952	13	7.09	5.05	0.01		
				5.922	~ 2	7.90	5.20	0.002		
				5.860	4	7.60	5.50	0.008		
				5.796	2	7.90	5.85	0.01		
				5.749	17	6.97	6.00	0.1		
				5.728	~ 1	8.21	6.15	0.008		
				5.704	15	7.03	6.25	0.2		
				5.651	~ 2	7.90	6.55	0.05		
Th^{228}		1.90	y	5.421	71	7.93	7.80	0.8	1	4, 53, b
				5.338	28	8.33	8.25	0.8	1.15	
				5.208*	0.4	10.18	9.00	0.06	0.089	
				5.173	0.2	10.48	9.20	0.05	0.074	
				5.137*	0.03	11.30	9.40	0.01	0.018	
Th^{229}		7340	y	5.02	~ 10	12.36	10.10	0.005		
				4.94	~ 20	12.06	10.60	0.03		
				4.85	~ 70	11.52	11.20	0.5		
Th^{230}		$8 \cdot 10^4$	y	4.682	75.5	12.52	12.30	0.6	1	54, b
				4.614	24.5	13.01	12.80	0.6	0.98	
				4.472	0.15	15.23	13.80	0.04	0.062	
				4.427*	0.03	15.92	14.15	0.02	0.028	
Th^{232}		$1.39 \cdot 10^{10}$	y	3.98		17.64	17.80	1.6		

TABLE 1 (continued).

Parent nucleus	% α	T_{tot}		E_α	Rel. α-int. (%)	$\log_{10} T$	$\log_{10} T_0$	F	c	References and remarks
$_{91}Pa^{226}$		1.8	m	6.81		2.03	1.70	0.5		
Pa^{227}	~ 85	38.3	m	6.46		3.43	3.20	0.6		
Pa^{228}	~ 2	22	h	6.09	75	6.72	4.80	0.01		
				5.85	25	7.20	6.00	0.06		
Pa^{231}		$3.43 \cdot 10^4$	y	5.0490	12	12.95	10.55	0.004		31, d
				5.0205	23	12.67	10.65	0.01		
				5.006	26	12.62	10.70	0.01		
				4.974	1.5	13.86	10.95	0.001		
				4.942	24	12.65	11.15	0.03		
				4.925	3	13.56	11.25	0.005		
				4.848	1.5	13.86	11.75	0.008		
				4.727	10	13.03	12.60	0.4	1	
				4.704	0.8	14.13	12.70	0.04		
				4.671	1.3	13.92	12.95	0.1	0.32	
				4.630	0.3	14.56	13.25	0.05		
$_{92}U^{227}$		1.3	m	6.8		1.89	2.20	2.0		
U^{228}	~ 80	9.3	m	6.67		2.84	2.70	0.8		
U^{229}	~ 20	58	m	6.42		4.24	3.75	0.3		
U^{230}		20.8	d	5.884	67.1	6.43	6.30	0.8	1	11, 42, b, e
				5.813	32.1	6.75	6.65	0.8	1.1	
				5.658*	0.4	8.65	7.50	0.08	0.091	
				5.658	0.4	8.65	7.50	0.08	0.091	
U^{231}	$5.5 \cdot 10^{-3}$	4.3	d	5.45		9.83	8.60	0.06		
U^{232}		74	y	5.318	68	9.54	9.35	0.6	1	9, 49, b
				5.261	32	9.86	9.75	0.8	1.0	
				5.134	0.32	11.86	10.50	0.04	0.059	
				4.998*	0.01	13.37	11.35	0.01	0.013	
U^{233}		$1.62 \cdot 10^5$	y	4.8157	83.5	12.79	12.55	0.6	1	31
				4.7732	14.9	13.53	12.80	0.2	0.34	
				4.7174	1.6	14.50	13.20	0.05	0.093	
				4.4890	0.03	14.23	14.85	0.04		
U^{234}		$2.5 \cdot 10^5$	y	4.763	75	13.02	12.95	0.8	1	31, 54
				4.710	25	13.50	13.25	0.6	0.74	
				4.593	0.3	15.42	14.10	0.05	0.062	
U^{235}		$7.1 \cdot 10^8$	y	4.58	10	17.35	14.20	0.0008		
				4.47	~ 3	17.87	15.00	0.001		
				4.40	83	16.43	15.50	0.1		
				4.20	4	17.75	17.15	0.3		
U^{236}		$2.4 \cdot 10^7$	y	4.50		14.88	14.85	1		
U^{238}		$4.5 \cdot 10^9$	y	4.18	78	17.26	17.30	1	1	54
				4.13	22	17.81	17.70	0.8	0.80	
				4.02	~ 0.1	20.15	18.70	0.03	0.033	

TABLE 1 (continued).

Parent nucleus	$^0/_0\ \alpha$	T_{tot}	E_α	Rel. α-int. ($^0/_0$)	$\log_{10} T$	$\log_{10} T_0$	F	c	References and remarks
$_{93}Np^{231}$		~ 50 m	6.28		3.48	4.85	24		
Np^{233}	$\sim 10^{-3}$	35 m	5.53		8.32	8.65	2.0		
Np^{235}	$3.5 \cdot 10^{-3}$	410 d	5.06		12.00	11.50	0.3		34
Np^{237}		$2.20 \cdot 10^6$ y	4.872	3.1	15.35	12.70	0.003		38
			4.816	3.5	15.30	13.10	0.006		
			4.787	53	14.12	13.25	0.1		
			4.767	29	14.38	13.40	0.1		
			4.713	1.7	15.61	13.80	0.02		
			4.674	3.3	15.32	14.05	0.05		
			4.644	6.0	15.06	14.30	0.2		
			4.589	0.5	16.14	14.70	0.04		
			4.52	0.02	17.54	15.20	0.005		
$_{94}Pu^{232}$	≥ 2	36 m	6.58		5.03	4.00	0.1		
Pu^{234}	~ 4	9 h	6.19		5.91	5.75	0.6		
Pu^{235}	~ 0.002	26 m	5.85		7.89	7.45	0.4		
Pu^{236}		2.7 y	5.75		7.93	7.95	1		
Pu^{238}		90 y	5.495	72	9.60	9.40	0.6	1	7, 31
			5.452	28	10.01	9.70	0.5	0.69	
			5.352	0.09	12.50	10.25	0.006	0.0087	
Pu^{239}		24360 y	5.150	69	12.05	11.50	0.3	1	31, 35, f
			5.137	20	12.59	11.55	0.1	0.33	
			5.099	11	12.84	11.80	0.1	0.35	
Pu^{240}		6300 y	5.162	76	11.42	11.35	0.8	1	24, 31, 35, 50
			5.118	24	11.92	11.70	0.6	0.62	
			5.014	0.1	14.30	12.35	0.01	0.012	
Pu^{241}	$\sim 5 \cdot 10^{-3}$	14 y	4.893	75	13.07	13.15	1.3		50, g
			4.848	25	13.55	13.50	1		
Pu^{242}		$\sim 5 \cdot 10^5$ y	4.898	80	13.30	13.15	0.8	1	50, h
			4.854	20	13.90	13.45	0.4	0.50	
$_{95}Am^{237}$	0.005	~ 1.3 h	6.01		7.97	7.10	0.1		
Am^{239}	0.003	12 h	5.75		9.16	8.45	0.2		
Am^{241}		470 y	5.5408	0.39	12.58	9.60	0.001		6, 31, i
			5.5082	0.24	12.79	9.85	0.001		
			5.4820	85	10.24	9.95	0.5	1	
			5.4391	12.84	11.06	10.25	0.2	0.31	
			5.3860	1.66	11.95	10.40	0.03	0.083	
			5.3210	0.015	14.00	10.95	0.001	0.0018	
			5.241	0.002	14.87	11.40	0.0003	0.00076	
Am^{243}		$7.6 \cdot 10^3$ y	5.340	0.17	14.15	10.85	0.0005		6, 51, j
			5.309	0.16	14.18	11.00	0.0006		
			5.267	87.1	11.44	11.30	0.8	1	
			5.224	11.5	12.32	11.55	0.2	0.25	
			5.169	1.1	13.34	11.90	0.04	0.052	

Table 1 (continued).

Parent nucleus	$^0/_0\ \alpha$	T_{tot}		$E\alpha$	Rel. α-int. ($^0/_0$)	$\log_{10} T$	$\log_{10} T_0$	F	c	References and remarks
$_{96}Cm^{238}$	> 10	2.5	h	6.50		4.95	5.20	1.6		
Cm^{240}		26.8	d	6.26		6.36	6.35	1		50
Cm^{241}	0.2	35	d	5.89		9.18	8.20	0.1		50
Cm^{242}		162	d	6.110	73.7	7.28	7.10	0.6	1	3, 50, b
				6.066	26.3	7.73	7.30	0.4	0.60	
				5.965	0.035	10.60	7.85	0.002	0.0028	
Cm^{243}		35	y	5.985	6	10.26	7.70	0.003		3, 42, 50, k
				5.777	81	9.13	8.85	0.5	1	
				5.732	13	9.93	9.10	0.2	0.29	
				5.675	~ 2	10.74	9.40	0.05		
Cm^{244}		19	y	5.798	75	8.90	8.75	0.8	1	3, 50
				5.755	25	9.38	8.90	0.3	0.56	
Cm^{245}		$26 \cdot 10^3$	y	5.34		11.91	11.35	0.3		50, l
$_{97}Bk^{243}$	~ 0.1	4.6	h	6.72	30	7.74	4.65	0.0008		
				6.55	53	7.49	5.45	0.01		
				6.20	17	7.99	7.05	0.1		
Bk^{245}	0.105	4.95	d	6.37	33	9.09	6.30	0.002		39
				6.17	41	9.00	7.30	0.02		
				5.89	26	9.19	8.60	0.3		
Bk^{249}	~ 0.001	290	d	5.40		12.40	11.50	0.1		37
$_{98}Cf^{244}$		~ 25	m	7.17		3.18	3.20	1		42
Cf^{245}		44	m	7.11		3.42	3.45	1		42
Cf^{246}		35.7	h	6.753	78	5.22	5.05	0.6	1	36
				6.711	22	5.77	5.20	0.3	0.44	
				6.608	> 0.08	8.21	5.65	0.003	0.0046	
Cf^{248}		205	d	6.26		7.25	7.30	1.3		50
Cf^{249}		500	y	6.19	3	11.72	7.60	0.00008		29
				5.93	6	11.42	9.00	0.004		
				5.80	90	10.24	9.70	0.3		
Cf^{250}		~ 10	y	6.024	83	8.58	8.55	1	1	10
				5.980	17	9.27	8.70	0.3	0.35	
Cf^{252}		~ 2	y	6.112	84.5	7.87	8.05	1.6	1	10
				6.069	15.5	8.61	8.25	0.4	0.30	
$_{99}E^{253}$		19.3	d	6.633	90.2	6.27	6.00	0.5	1	42
				6.592	7.7	7.34	6.10	0.06	0.13	
				6.545	1.8	7.97	6.30	0.02	0.050	
				6.493	0.3	8.74	6.60	0.008		
				6.25	0.05-0.1	9.35	7.80	0.03		
$_{100}Fm^{252}$		22.7	h	7.04		4.91	4.60	0.5		26
Fm^{254}		3.3	h	7.17	85	4.14	4.05	0.8	1	8, 25
				7.128	15	4.90	4.15	0.2	0.26	
				7.034	> 0.2	6.77	4.60	0.006	0.0085	

Remarks to Table 1.

a) Briggs[19] gives quite different data for Ra^{223}.

b) The odd parity alpha groups of even-even nuclei are recognized by asterisks * placed after their $E\alpha$-values. The assignment 1– for the lowest odd parity state of the daughter nucleus is certain for the parent nuclei Th^{226}, Th^{228}, Th^{230}, U^{230}, and almost sure for the parent nucleus U^{232}. In the alpha decay of Cm^{242}, one has also found a $0+ \rightarrow 1-$ alpha transition (not listed here), but, in this case, the 1– interpretation is somewhat doubtful[52]. See also Note added in proof, p. 46.

c) The data on Th^{227} reported by Frilley et al.[27] are the same as those on Th^{227} in HPS which are listed first, but Briggs'[19] data are different.

d) The two alpha groups of energies 4.925 MeV and 4.630 MeV were not reported by Goldin et al.[31]; they are obtained by Hummel (private communication from Professor Perlman).

e) The two alpha groups of 5.658 MeV energy have actually not been resolved, but beta decay data indicate that about half of the 5.658 MeV alpha particles leave the daughter nucleus in the 4+ state and the other half in the 1– state (private communication from Professor Perlman).

f) The most energetic alpha group which has been observed for the parent nucleus Pu^{239} (i. e., the group with the alpha particle energy 5.150 MeV) does not appear to correspond to a transition to the ground state of the daughter nucleus[2].

g) The decay energy of Pu^{241} (calculated from closed cycles) is 5.13 MeV, according to Glass et al.[30]. The most energetic alpha group which has been observed for the parent nucleus Pu^{241} (i. e., the group with the alpha particle energy 4.893 MeV) has a transition energy which is less than the decay energy by about 0.150 MeV, and therefore it seems probable that this alpha group does not leave the daughter nucleus in its ground state.

h) For the half-life of Pu^{242}, we have used the value $5 \cdot 10^5$ years, which is given in HPS. The value $9 \cdot 10^5$ years given by Seaborg and Katz[50] is too high. New measurements have yilded a value of about $4 \cdot 10^5$ years [39a, 42].

It should also be remarked that the measurement (by Asaro) from which the values 80 % and 20 % for the relative intensities of the two alpha groups of the parent nucleus Pu^{242} are obtained is very rough (private communication from Professor Perlman).

i) There may be some uncertainty about the 5.241 MeV alpha group.

j) On the basis of experimental indication for a ground state spin 1/2 for Np^{239}, objection has been raised by Stephens et al.[51] against the assignment of the spin 5/2 to the level in which the alpha group with the alpha particle energy 5.267 MeV leaves the daughter nucleus Np^{239}. The assignment of spin 1/2 for the ground state of Np^{239} implies, however, many difficulties for the interpretation of the level structure of Np^{239} as well as the beta decay of this nucleus [35a].

k) The 5.985 MeV alpha group from the parent nucleus Cm^{243} leaves the daughter nucleus in an excited state which lies 76 keV above the ground state [43a].

The state in which the 5.675 MeV alpha group leaves the daughter nucleus decays to the lowest state of the favoured band (in which the 5.777 MeV alpha group leaves the daughter nucleus) by an $E1$ gamma transition, and therefore the first of these states probably consists of two different states lying close together (private communication from Professor Perlman). This fact modifies an interpretation which has recently been given by A. Bohr et al.[15].

l) For the parent nucleus Cm^{245}, the data given by Seaborg and Katz[50] are more recent and probably more accurate than those given in HPS. It should also be remarked that alpha-gamma coincidence experiments indicate that the 5.34 MeV alpha group does not leave the daughter nucleus in its ground state (private communication from Professor Perlman).

Table 2.*

For the alpha groups of even-even nuclei, which are characterized by their angular momenta l, we here list the empirical hindrance factors which we have obtained as private communication from Professor Perlman. The reduced transition probabilities c_l have been calculated as the reciprocals of these hindrance factors (except for the alpha group α_3 of the parent nucleus Th^{223} for which c_3 has been obtained from Table 1). From the c_l-values we have then calculated the b_l-values according to (VIII-7) and (VIII-12).

Parent nucleus	l	Hindrance factor	c_l	b_l
$_{88}Ra^{222}$	2	1.20	0.83	1.18
Ra^{224}	2	1.20	0.83	1.19
Ra^{226}	2	0.90	1.11	1.38
$_{90}Th^{226}$	2	1.50	0.67	1.06
	1	4.0	0.25	0.55
	4	7.8	0.13	0.84
Th^{228}	2	0.90	1.11	1.37
	1	10	0.10	0.35
	4	12	0.083	0.69
	3		0.018	0.23
Th^{230}	2	1.0	1.00	1.30
	4	9.8	0.10	0.77
	1	40	0.025	0.17
	6	> 8500	< 0.00012	< 0.070
Th^{232}	2	0.80	1.25	1.47
$_{92}U^{230}$	2	1.1	0.91	1.23
	1	11	0.091	0.33
	4	9.6	0.10	0.76
U^{232}	2	1.1	0.91	1.24
	4	16	0.062	0.60
	1	70	0.014	0.13
	6	> 270	< 0.0037	< 0.37
U^{234}	2	1.2	0.83	1.18
	4	14.5	0.069	0.62
U^{236}	2	1.1	0.91	1.24

Parent nucleus	l	Hindrance factor	c_l	b_l
U^{238}	2	1.5	0.67	1.06
$_{94}Pu^{236}$	2	1.5	0.67	1.05
Pu^{238}	2	1.6	0.62	1.02
	4	115	0.0087	0.22
	6	385	0.0026	0.30
	8	15000	0.000067	0.18
Pu^{240}	2	1.7	0.59	0.99
	4	68	0.015	0.28
$_{96}Cm^{242}$	2	1.65	0.61	1.00
	4	380	0.0026	0.12
	6	260	0.0038	0.36
	8	4850	0.00021	0.28
	1	1000	0.0010	0.035
Cm^{244}	2	1.8	0.56	0.96
	4	1000	0.0010	0.073
	6	340	0.0029	0.31
$_{98}Cf^{246}$	2	2.7	0.37	0.77
	4	115	0.0087	0.21
	6	320	0.0031	0.31
Cf^{250}	2	3.0	0.33	0.73
Cf^{252}	2	3.2	0.31	0.71
	4	~ 94	~ 0.011	~ 0.23
$_{100}Fm^{254}$	2	4.0	0.25	0.63
	4	54	0.018	0.30
	6	> 800	< 0.0012	< 0.19

* Note added in proof: Somewhat more recent data are given in the review article on alpha decay by Perlman and Rasmussen[43a] (see their fig. 15) which will appear in Handbuch der Physik. To our purpose, however, the data in Table 2 can be used equally well.

In Table 2 we list empirical values of the hindrance factors for the alpha groups of even-even nuclei, kindly communicated to us by Professor Perlman. This table contains some hindrance factors for alpha groups populating higher states of the daughter nucleus which cannot be obtained from Table 1. The reduced transition probabilities c_l in Table 2 have been calculated as the reciprocals of the hindrance factors listed there (except for the alpha group α_3 of Th^{223} for which c_3 has been obtained from Table 1). With a few exceptions, the c_l-values in Table 2 are in satisfactory agreement with those in Table 1. The exceptions may be associated with differences in the definitions used for the hindrance factors and their reciprocals, which are the reduced transition probabilities c_l. For the discussion of the empirical data in relation to the present theory, which will be given in the next chapter, it is better to use not the reduced transition probabilities c_l themselves, but certain quantities b_l which are defined by (VIII-12). The b_l-values which are listed in Table 2 have been calculated from the c_l-values there by assuming the nuclear radii to be given by (VIII-7).

VIII. Comparison between Theory and Empirical Data.

In this chapter, we first give a brief review of the theoretical treatment developed in the Chapters III–VI, and subsequently discuss the ability of this theory to account for the available empirical data.

To treat the alpha fine structure problem we must describe the penetration of the alpha particle through the potential barrier around the deformed daughter nucleus by means of a wave function which takes into account the possibility for the alpha particle to leak out in different channels corresponding to different final states of the daughter nucleus. Using the center-of-mass coordinate system for the alpha particle and the daughter nucleus we have constructed this wave function in such a way that its total angular momentum is I_i and its angular momentum component along a fixed direction in space is M_i, where I_i and M_i denote the corresponding quantum numbers for the parent nucleus. This wave function contains for each channel a function which depends on the distance r of the alpha particle from the center of the daughter nucleus and which will be referred to as the radial channel function. These channel functions satisfy a system of coupled differential equations, which has been derived on the assumption that the daughter nucleus possesses axial symmetry and can be described in terms of a rotational motion superimposed on the intrinsic motion of the nucleons in the deformed nuclear potential. If the small coupling causing transitions between different rotational bands of the daughter nucleus during the alpha particle penetration through the potential barrier is neglected, the differential equations couple only those radial channel functions which correspond to alpha groups which emerge from a given state (K_i, I_i) of the parent nucleus and leave the daughter nucleus in the same rotational band characterized by the quantum number K_f. In terms of the alpha wave function on the nuclear surface, these differential equations determine the transition probabilities for the different alpha groups.

The passage of the alpha particle through the region near the nuclear surface, where appreciable exchange of angular momentum occurs between the alpha particle and the daughter nucleus, takes place in a time which is short compared to the nuclear rotational period. Therefore it is in first approximation allowed to consider the nucleus as fixed in space, while the alpha particle traverses this interior region. An approximate solution of the problem concerning the penetration of an alpha particle through such a potential barrier fixed in space can be obtained by means of an approximation procedure which has been described in a qualitative way in Chapter VI, and which is developed more rigorously in Appendix B. Using this approximation method, one finds that in the interior part of the potential barrier of a deformed nucleus the alpha wave function can in first approximation be calculated from the alpha wave function on the nuclear surface by means of a formula which represents a generalization of the well-known formula for the spherically symmetric potential barrier.

In the outer part of the potential barrier and outside of this barrier the Coulomb field around the daughter nucleus is almost spherically symmetric, and there is then practically no coupling between the different channels into which the alpha particle is able to leak out. Therefore, the passage of the alpha particle through this outer region takes place approximately in the same way as the passage of an alpha particle of given energy through a spherically symmetric potential barrier, the alpha transition energy being that corresponding to the channel considered. Except for constant factors, one can therefore in the outer region easily find the radial channel functions which correspond to outgoing waves.

Assuming that there exists an intermediate region around a spherical surface of radius R_1, where the above approximations for the interior and exterior regions both apply, we can determine the unknown constant factors which appear in our approximate solution for the outer region by matching this solution to that for the interior region at the spherical surface $r = R_1$. In this way, the radial channel functions are determined in the whole space outside the nuclear surface. They depend, of course, on the boundary conditions for the alpha wave function on the nuclear surface.

By applying the approximation procedure outlined above to the alpha transition from the state (I_i, K_i) of a given parent nucleus to the state (I_f, K_f) of its daughter nucleus, we have in Chapter VI found that, for $r \geqslant R_1$, the radial channel function $f_{K_f, l, I_f}(r)$ corresponding to the angular momentum l is given by (VI-12), the transition probability P_{K_f, I_f} is given by (VI-13), and the angular distribution of the alpha group considered is given by (VI-14). Thus it is seen that the alpha decay depends on a quantity B, defined by (VI-9), which is a measure of the quadrupole deformation, and on the expansion coefficients a_l of the function ψ_1, defined by (VI-10), which depends on the alpha wave function on the nuclear surface, ψ_0, as well as on the higher multipoles of the nuclear surface. The further dependence on the transition energy, the atomic number, the nuclear radius, and the quantum numbers K_f, I_i, and I_f should also be mentioned.

Instead of considering the transition probabilities it is convenient to introduce the so-called reduced transition probabilities in which the strong energy dependence has been separated off. In the definition used in Chapter VII, this energy dependence has been taken directly from the empirical data on the ground state alpha transitions of even-even nuclei. This definition of the reduced transition probabilities is convenient when one deals with the empirical data, but in more theoretical connections it is often advantageous to choose a slightly different definition according to which the energy depending factor to be separated off is taken from the well-known theory for the penetration of an alpha particle of angular momentum zero through a spherically symmetric potential barrier. For practical purposes, it is in general unnecessary to distinguish between these two definitions, for they give approximately the same numerical results, unless the energy differences of the alpha groups considered are too large.

In the following sections, we chiefly discuss the alpha decays of even-even nuclei and the favoured alpha decays of odd-A nuclei. For these alpha decays the alpha particle is formed with $\Omega = 0$. In order to simplify the notations we shall, therefore, in general write $k_{l,l'}(B)$ instead of $k^0_{l,l'}(B)$ and a_l instead of $a_{l,0}$.

a. Ground state alpha transition in even-even nuclei.

According to Chapter VI, the alpha decay of an actual nucleus takes place in the same way as that of a fictive nucleus with the purely ellipsoidal surface

$$R(\vartheta') = R_0 \left(1 + \beta_2 Y_{2,0}(\vartheta')\right) \qquad \text{(VIII-1)}$$

on which the alpha wave function is equal to $\psi_1(\vartheta')$ and outside of which the electrostatic potential is assumed to be equal to the sum of the spherically symmetric potential and the pure quadrupole potential of the actual nucleus. The angular dependence of $\psi_1(\vartheta')$ is due both to the higher multipoles in the nuclear deformation and to the variation of the alpha wave function on the nuclear surface. For the ground state alpha transition probability of an even-even nucleus we obtain

$$P_{0,0} = v_0 \left(\frac{R_0}{G_0(E_0, R_0)}\right)^2 \left|\sum_l k_{0,l}(B)\, a_l\right|^2 \qquad \text{(VIII-2)}$$

as a special case of (VI-13). The quantities a_l appearing here are the coefficients in the expansion of $\psi_1(\vartheta')$ in terms of the spherical harmonics $Y_{l,0}(\vartheta')$. From (VIII-2) we find that, if we assume $\psi_1(\vartheta')$ to be independent of ϑ' and to remain constant while β_2 varies, the ellipsoidal deformation of the nucleus causes the ground state alpha transition probability to be enhanced by the factor*

$$(k_{0,0}(B))^2 = \left(\int e^{B P_2(\cos\vartheta)} \frac{d\omega}{4\pi}\right)^2 = \left(\int_0^1 e^{B P_2(x)}\, dx\right)^2 \qquad \text{(VIII-3)}$$

* The enhancement factor (VIII-3) has been obtained on the assumption that R_0 remains constant as β_2 varies. There is a slight difference between this enhancement factor and that corresponding to constant nuclear volume for varying β_2, but for our purpose the difference is unimportant.

which increases as $|B|$ increases, i. e., as $|\beta_2|$ increases. HILL and WHEELER[33], who were the first to point at the importance of the nuclear deformation for alpha decay, estimated the enhancement very roughly and obtained an expression which in our notations reads

$$\underset{\vartheta}{\mathrm{Max}}\, e^{2\,B\,P_2\,(\cos\vartheta)}\,; \qquad \text{(VIII-4)}$$

however, this expression overestimates the enhancement considerably*. By means of a straightforward improvement of the considerations used by HILL and WHEELER one may replace their expression for the enhancement by

$$\int e^{2\,B\,P_2\,(\cos\vartheta)} \frac{d\omega}{4\,\pi} = \int_0^1 e^{2\,B\,P_2\,(x)}\, dx = k_{0,\,0}\,(2\,B), \qquad \text{(VIII-5)}$$

but also this expression overestimates the effect, as is easily seen by comparing (VIII-5) with (VIII-3) and using SCHWARZ's inequality.

For different values of B, we give in Table 3 the deformation parameter β_2, the intrinsic quadrupole moment Q_0, and the value of the enhancement factor according

TABLE 3. Calculated values of the enhancement factor for different nuclear deformations and in different degrees of approximation.

Q_0 ($10^{-24}\,\mathrm{cm}^2$)	β_2	B	Enhancement factor according to		
			(VIII-3)	(VIII-5)	(VIII-4)
– 18.8	– 0.35	– 3	3.49	5.93	20.1
– 12.5	– 0.24	– 2	1.88	2.67	7.4
– 6.3	– 0.12	– 1	1.20	1.37	2.7
0	0	0	1	1	1
6.3	0.12	1	1.24	1.55	7.4
12.5	0.24	2	2.41	5.11	54.6
18.8	0.35	3	6.93	23.98	403.4

to the correct formula (VIII-3) as well as according to (VIII-4) and (VIII-5). The values of β_2 have been obtained from the approximate formula

$$B \approx 8.5\,\beta_2 \qquad \text{(VIII-6)}$$

which is found to be fulfilled to within 10 %. This formula follows from (VI-9) and the empirical alpha decay data for even-even nuclei, if the nucleus is assumed to be

* On the basis of numerical calculations, the increase in the alpha decay rate due to the nuclear deformation has been discussed by RASMUSSEN and SEGALL[45] who obtained much smaller enhancement factors than HILL and WHEELER[33]. Their results seem to be consistent with ours, but a detailed comparison is difficult because of different boundary conditions for the alpha wave function.

uniformly charged (i. e., $q_0 = 1$) and the radius R_0 of the daughter is assumed to be given by

$$R_0 = 1.44 \cdot 10^{-13} (A-4)^{\frac{1}{3}} \mathrm{cm}, \tag{VIII-7}$$

where A is the mass number of the parent nucleus. The values of the intrinsic quadrupole moment Q_0 given in Table 3 have been calculated from (VI-2) and (VIII-7) by assuming that $q_0 = 1$, $Z-2 = 90$, and $A-4 = 230$.

There is a certain ambiguity in the choice of the radius R_0, and there may be some reasons for choosing R_0 somewhat smaller than according to (VIII-7), e. g.,

$$R_0 = 1.2 \cdot 10^{-13} (A-4)^{\frac{1}{3}} \mathrm{cm}. \tag{VIII-8}$$

The choice of such a smaller radius would, however, not change the qualitative conclusions which we shall arrive at in this chapter. The larger radius (VIII-7) appears to represent the extension of the range of interaction between the alpha particle and the daughter nucleus[13]. For the estimate of β_2 from the electric quadrupole moment Q_0 a somewhat smaller radius should be employed. With the present accuracy of the Q_0-values this point is, however, of minor significance.

Empirically it is known that the probabilities for the ground state alpha transitions of even-even nuclei can be represented with good approximation (within a factor of about 2) by a smoothed out function $P_0(Z, E_0)$ which depends only on the atomic number Z of the parent nucleus and on the alpha decay energy E_0. This function can easily be obtained from the formula

$$P_0 = \frac{ln\, 2}{T_0}, \tag{VIII-9}$$

where T_0 is the smoothed out half-life function (depending on the atomic number Z of the parent nucleus and on the alpha particle energy E_α) which has been determined in a semi-empirical manner in Chapter VII. The variation of the enhancement factor with the nuclear deformation should obviously affect the function $P_0(Z, E_0)$. From Coulomb excitation data it has been found that $Q_0 \sim 10$ for the heavy elements around U. (See the review article by Alder et al.[1].) For these Q_0-values the correct enhancement factor (calculated from (VIII-3)) is about 2 according to Table 3. A factor in $P_0(Z, E_0)$ of at most this order of magnitude and varying slowly with Z and A is difficult to separate from other causes for dependence of P_0 on Z and A and from experimental uncertainties. From this fact and the assumption that the alpha particle formation probability does not depend especially strongly on the nuclear deformation, we obtain an explanation of the very reasonable values which one obtains for the nuclear radii from $P_0(Z, E_0)$ by means of the simple Gamow theory. With the same assumption, the large factor (VIII-4) estimated by Hill and Wheeler[33] and amounting to ~ 40 (cf. Table 3) should have been easily detectable in the behaviour of the function $P_0(Z, E_0)$ and the nuclear radii calculated from it in the usual way.

b. Even parity alpha transitions in even-even nuclei.

Most of the alpha groups which have been found in even-even nuclei are of the even parity type for which the nuclear transitions are $0+ \rightarrow 0+, 2+, 4+, \ldots$. The alpha groups corresponding to these transitions are denoted by $\alpha_0, \alpha_2, \alpha_4, \ldots$, respectively. The transition probability $P_{0,l}$ for the group α_l is given by the formula

$$P_{0,l} = v_l \left(\frac{R_0}{G_0(E_l, R_0)} \right)^2 \left| \sum_{l'} k_{l,l'}(B) a_{l'} \right|^2 \exp\left\{ -\frac{2\,l(l+1)}{\varkappa} \sqrt{\frac{\varkappa}{k R_0} - 1} \right\}, \qquad \text{(VIII-10)}$$

which is obtained by specializing (VI-13) to the case that $I_i = K_f = 0$ (which implies that the alpha wave function on the nuclear surface contains only the angular momentum component $\Omega = 0$). Instead of using $P_{0,l}$ itself, we prefer to use the corresponding reduced transition probability c_l which we define as the quotient of $P_{0,l}(E_l)$ and $P_{0,0}(E_l)$, where the explicitly denoted dependence on E_l indicates that not only $P_{0,l}$ but also $P_{0,0}$ shall be calculated for the transition energy E_l of the group α_l. From the formula (VIII-10) we then get

$$c_l = \left| \frac{\sum_{l'} k_{l,l'}(B) a_{l'}}{\sum_{l'} k_{0,l'}(B) a_{l'}} \right|^2 \exp\left\{ -\frac{2\,l(l+1)}{\varkappa} \sqrt{\frac{\varkappa}{k R_0} - 1} \right\}, \qquad \text{(VIII-11)}$$

if we note that it is only through the function $G_0(E_l, R_0)$ that the expression (VIII-10) for $P_{0,l}$ depends essentially on E_l. It is useful to carry the reduction of the alpha transition probabilities for even-even nuclei one step further and to separate away not only the energy dependence, but also the l-dependence which is due to the ordinary centrifugal barrier. To this purpose, we introduce, instead of c_l, a quantity b_l which we define by

$$b_l = \sqrt{c_l} \exp\left\{ \frac{l(l+1)}{\varkappa} \sqrt{\frac{\varkappa}{k R_0} - 1} \right\}. \qquad \text{(VIII-12)}$$

From this definition and the theoretical formula (VIII-11) for c_l we then get

$$b_l = \left| \frac{\sum_{l'} k_{l,l'}(B) a_{l'}}{\sum_{l'} k_{0,l'}(B) a_{l'}} \right|. \qquad \text{(VIII-13)}$$

For the quantities $k_{l,l'}(B)$ corresponding to even values $(0, 2, 4, 6, 8, \ldots)$ of both l and l' we give the following numerical results, written in matrix form:

$$\begin{array}{cccccc} l=0 & l=2 & l=4 & l=6 & l=8 & \end{array}$$

$$k(-3) = \left\{ \begin{array}{rrrrrr} 1.87 & -1.41 & 0.64 & -0.21 & 0.06 & \cdots \\ -1.41 & 1.51 & -1.02 & 0.47 & -0.16 & \cdots \\ 0.64 & -1.02 & 1.30 & -0.95 & 0.45 & \cdots \\ -0.21 & 0.47 & -0.95 & 1.28 & -0.95 & \cdots \\ 0.06 & -0.16 & 0.45 & -0.95 & 1.29 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right\}$$

$$k(-2) = \left\{ \begin{array}{rrrrrr} 1.37 & -0.84 & 0.28 & -0.07 & 0.01 & \cdots \\ -0.84 & 1.07 & -0.61 & 0.21 & -0.05 & \cdots \\ 0.28 & -0.61 & 0.99 & -0.60 & 0.21 & \cdots \\ -0.07 & 0.21 & -0.60 & 0.99 & -0.60 & \cdots \\ 0.01 & -0.05 & 0.21 & -0.60 & 1.00 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right\}$$

$$k(-1) = \left\{ \begin{array}{rrrrrr} 1.09 & -0.41 & 0.07 & -0.01 & 0.00 & \cdots \\ -0.41 & 0.89 & -0.32 & 0.06 & -0.01 & \cdots \\ 0.07 & -0.32 & 0.89 & -0.31 & 0.06 & \cdots \\ -0.01 & 0.06 & -0.31 & 0.89 & -0.31 & \cdots \\ 0.00 & -0.01 & 0.06 & -0.31 & 0.89 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right\}$$

$k(0)$ = the unit matrix, i. e., $k_{l,l'}(0) = \delta_{l,l'}$

$$k(1) = \left\{ \begin{array}{rrrrrr} 1.11 & 0.55 & 0.11 & 0.01 & 0.00 & \cdots \\ 0.55 & 1.56 & 0.54 & 0.10 & 0.01 & \cdots \\ 0.11 & 0.54 & 1.49 & 0.53 & 0.10 & \cdots \\ 0.01 & 0.10 & 0.53 & 1.48 & 0.52 & \cdots \\ 0.00 & 0.01 & 0.10 & 0.52 & 1.48 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right\}$$

$$k(2) = \left\{ \begin{array}{rrrrrr} 1.55 & 1.53 & 0.59 & 0.15 & 0.03 & \cdots \\ 1.53 & 3.03 & 1.77 & 0.60 & 0.14 & \cdots \\ 0.59 & 1.77 & 2.82 & 1.67 & 0.58 & \cdots \\ 0.15 & 0.60 & 1.67 & 2.76 & 1.65 & \cdots \\ 0.03 & 0.14 & 0.58 & 1.65 & 2.76 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right\}$$

$$k(3) = \left\{ \begin{array}{rrrrrr} 2.63 & 3.56 & 1.95 & 0.71 & 0.20 & \cdots \\ 3.56 & 6.58 & 4.79 & 2.23 & 0.76 & \cdots \\ 1.95 & 4.79 & 6.17 & 4.46 & 2.10 & \cdots \\ 0.71 & 2.23 & 4.46 & 5.95 & 4.37 & \cdots \\ 0.20 & 0.76 & 2.10 & 4.37 & 5.93 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right\}$$

$$\begin{array}{cccccc} l=0 & l=2 & l=4 & l=6 & l=8 & \end{array}$$

From the empirical data on the alpha decays of even-even nuclei, Perlman and co-workers obtained the hindrance factors which are listed in Table 2. In this table, we have also listed the corresponding empirical c_l-values which are the reciprocals of the hindrance factors, and the b_l-values which we have obtained from the empirical c_l-values according to (VIII-12), assuming R_0 to be given by (VIII-7). The empirical knowledge thus obtained about the b_l-values for the even parity alpha groups of even-even nuclei is summarized in Table 4. It is seen that b_2 decreases as Z increases, b_4 has a sharp minimum for $Z = 96$, and b_6 has a broad maximum at about $Z = 96$. Also b_8 is rather large for this Z-value, but there is not much known about its variation with Z. We shall now discuss this empirical evidence in connection with

Table 4. Survey of the empirical b_l-values for the even parity alpha groups of even-even nuclei.

Parent nucleus	b_2	b_4	b_6	b_8
Ra	1.2 – 1.4			
Th	1.0 – 1.5	0.7 – 0.9	< 0.1	
U	1.0 – 1.3	0.6 – 0.8	< 0.4	
Pu	1.0	0.2 – 0.3	0.3	0.2
Cm	1.0	0.1	0.3 – 0.4	0.3
Cf	0.7 – 0.8	0.2	0.3	
Fm	0.6	0.3	< 0.2	

the theoretical formula (VIII-13). From this formula it is seen that the b_l-values for the even parity alpha groups depend only on the coefficients $a_0, a_2, a_4, \ldots$ and on the matrix elements $k_{l,l'}(B)$ corresponding to even values of l and l'. Because of the appearance of the sign for the absolute value on the right-hand side of (VIII-13), it is also clear that, even if B is assumed to be known, one cannot determine $a_0, a_2, a_4, \ldots$ uniquely from the empirical b_l-values (for even l). Therefore it seems most reasonable to proceed by making simple assumptions about the shape of the nuclear surface and the alpha wave function on this surface, and to see to what extent one can understand the empirical data on the basis of such assumptions. According to the formulae in Chapter VI, it is not possible to separate the effect of the nuclear shape from that of the alpha wave function on the nuclear surface, ψ_0. If ψ_0 is assumed not to change sign, it is also seen that this function can be replaced by a constant if at the same time the values of the actual deformation parameters β_λ are replaced by certain effective values. In the following, we therefore assume ψ_0 to be constant, but we must remember the effective nature of the nuclear deformation parameters β_λ. We shall now make some simple assumptions about the effective nuclear shape and discuss the b_l-values thus obtained in relation to the empirical b_l-values.

We first consider the simple assumption that the shape of the nuclear surface is due to a pure quadrupole deformation and that the alpha wave function on this surface is constant. The coefficients a_l are then equal to zero for $l \neq 0$, and from (VIII-13) and the numerical values of $k_{l,0}(B)$ for even l we get the b_l-values which are listed in Table 5 for different values of B. The β_2-values corresponding to these B-values have been obtained by means of the formula (VIII-6). From Q_0-values obtained from Coulomb excitation data it follows that, for the nuclei around Th and U, the actual deformation parameter β_2 has values of about 0.25. Empirically there is not much known about the variation of β_2 with Z in the region of the very heavy elements, but for the moment we shall assume that β_2 increases slightly with Z. Hence, we shall compare the empirical b_l-values in Table 4 with those theoretical b_l-values in Table 5 which correspond to β_2-values lying in the neighbourhood of 0.25 and

TABLE 5. Theoretical values of b_2, b_4, b_6, and b_8 for the case that the nuclear surface possesses a pure quadrupole deformation and the alpha wave function is constant on this surface.

β_2	B	b_2	b_4	b_6	b_8
− 0.35	− 3	0.76	0.34	0.11	0.03
− 0.24	− 2	0.61	0.20	0.05	0.01
− 0.12	− 1	0.38	0.07	0.01	0.00
0	0	0	0	0	0
0.12	1	0.49	0.10	0.01	0.00
0.24	2	0.98	0.38	0.09	0.02
0.35	3	1.35	0.74	0.27	0.08

increasing slightly with Z. The calculated b_2-values are of the right order of magnitude, but their slight variation with β_2 is opposite to and less than what we should expect from the variation of the empirical b_2-values with Z. The calculated b_4-values have very roughly the right order of magnitude, but do not show the observed variation with Z. In fact, the calculated b_4-values increase slightly with β_2, whereas the empirical b_4-values change fairly rapidly with Z and have a sharp maximum for $Z = 96$. The calculated b_6-values increase slightly with β_2, whereas the empirical b_6-values have a broad maximum at about $Z = 96$. Furthermore, the calculated b_6-values are smaller than the largest empirical b_6-values. The b_8-values are empirically known only for two isotopes having the atomic numbers $Z = 94$ and $Z = 96$, and these b_8-values are considerably larger than the small b_8-values which we have calculated. Summarizing these results we see that, on the basis of the above simple assumption, we get the right order of magnitude for b_2, whereas the higher b_l-values deviate the more from the corresponding empirical values, the higher l becomes. Furthermore, the empirically found variations of b_l with Z are not reproduced by the calculated b_l-values for any l.

We next assume the shape of the nuclear surface to be due to both a P_2-deformation and a P_4-deformation. The alpha wave function on the nuclear surface is still assumed to be constant. In a similar way as we have previously characterized the P_2-deformation by the quantity B, defined by (VI-9), we now characterize the P_4-deformation by a quantity C which we define by

$$C = \varkappa \left(\beta_4 \sqrt{\frac{k R_0}{\varkappa}} \left(1 - \frac{k R_0}{\varkappa}\right) - \frac{2\pi}{9(Z-2)} \left(\frac{k}{\varkappa}\right)^4 \frac{Q_4'}{\varepsilon} \int_{\frac{k R_0}{\varkappa}}^{1} \frac{dx}{x^4 \sqrt{x(1-x)}} \right). \quad \text{(VIII-14)}$$

In the parenthesis on the right-hand side of this expression we may approximately neglect the second term. By using (VI-9) and (VIII-6) we then get the following approximate connection between C and β_4:

$$C \approx \frac{2B}{\beta_2} \beta_4 \approx 17 \beta_4. \quad \text{(VIII-15)}$$

According to (VI-10) and (VIII-14), we can express $\psi_1(\vartheta)$ in terms of C as follows:

$$\psi_1(\vartheta) = \psi_0 \, e^{C Y_{4,0}(\vartheta)} = \psi_0 \left\{1 + C Y_{4,0}(\vartheta) + \tfrac{1}{2} C^2 (Y_{4,0}(\vartheta))^2 + \ldots\right\} = \sum_l a_l \, Y_{l,0}(\vartheta),$$

where

$$\left.\begin{array}{l} a_0 = \psi_0 (3.54 + 0.14\, C^2 + \ldots) \\ a_2 = \psi_0 (0.08\, C^2 + \ldots \qquad\ ) \\ a_4 = \psi_0 (C + 0.07\, C^2 + \ldots \quad\ ) \\ a_6 = \psi_0 (0.07\, C^2 + \ldots \qquad\ ) \\ a_8 = \psi_0 (0.12\, C^2 + \ldots \qquad\ ) \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\} \quad \text{(VIII-16)}$$

For comparatively small values of C (e.g., for $C \lesssim 1$), the b_l-values can now be calculated by substituting the series expansions (VIII-16) into the theoretical formula (VIII-13). If β_2 is assumed to be equal to the actual deformation parameter β_2 which we, for the sake of simplicity, assume to have a constant value of about 0.24 (corresponding to $B = 2$) throughout the whole region of the very heavy nuclei*, and if β_4 ($\approx C/17$) is determined such that the calculated b_4-values agree with the empirical b_4-values, we find the result shown in Table 6. Actually, there exists more than one possible C-value for every Z-value, but we have more or less arbitrarily required that $|C|$ shall not exceed 1, and, on this assumption, the determination of the C-values (in Table 6) from the empirical b_4-values is unique. The magnitudes of the calculated b_2-values as well as their decrease with increasing Z in the region from Th to Cm are in reasonable agreement with the empirical b_2-values, but the slight increase with Z of the calculated b_2-values in the region from Cm to Fm is not in quite satisfactory agreement with the behaviour of the empirical b_2-values in this region. The calculated b_6-values differ from the empirical b_6-values both as regards the orders of magnitude of these values and their variation with Z. It is also seen that the calculated b_8-values

* For our later qualitative conclusions it is quite unessential whether β_2 is here considered to be constant or changing slightly with Z.

TABLE 6.

Values of C, β_4 ($\approx C/17$), b_2, b_6, and b_8 obtained by adjusting C such that the calculated b_4-values agree with the corresponding empirical values. Here B has been assumed to be constant and equal to 2 (which corressponds to $\beta_2 \approx 0.24$) for all the nuclei considered.

Parent nucleus	b_4 (emp.)	C	β_4	b_2	b_6	b_8
Th	0.8	0.90	0.053	1.18	0.39	0.16
U	0.7	0.67	0.039	1.14	0.31	0.12
Pu	0.25	− 0.28	− 0.016	0.92	0.02	0.00
Cm	0.1	− 0.64	− 0.038	0.86	0.06	0.01
Cf	0.2	− 0.40	− 0.024	0.90	0.01	0.01
Fm	0.3	− 0.16	− 0.009	0.95	0.05	0.00

for the parent nuclei of Pu and Cm are appreciably too low. By considering both β_2 ($\approx B/8.5$) and β_4 ($\approx C/17$) as adjustable parameters, and determining them such that the calculated values of b_2 and b_4 shall agree with the corresponding empirical values, we have found the figures in Table 7. The table shows that the effective deformation parameter β_2 is constant in the region of daughter nuclei between Ra and Pu, but decreases with increasing atomic number in the region of daughter nuclei above Pu. In connection with this result it may be noted that calculations of the nuclear quadrupole deformations indicate that the actual deformation parameter β_2 increases slightly as one goes from Ac to Am, but may perhaps begin to decrease with increasing atomic number in the region above Am (private communication from Dr. NILSSON). It must, however, be borne in mind that our effective deformation parameter β_2 may differ from the corresponding actual deformation parameter for the nuclear surface. The b_6- and b_8-values in Table 7 behave essentially in the same way as the corresponding values in Table 6, and they deviate considerably from the empirical values. Hence, it is not sufficient to assume only the two deformation parameters β_2 and β_4

TABLE 7. Result of adjusting the parameters B ($\approx 8.5\,\beta_2$) and C ($\approx 17\,\beta_4$) such that the calculated values of b_2 and b_4 shall agree with the corresponding empirical values.

Parent nucleus	B	β_2	C	β_4	b_2	b_4	b_6	b_8
Th	2.2	0.26	0.7	0.041	1.28	0.78	0.39	0.14
U	2.2	0.26	0.5	0.029	1.24	0.68	0.32	0.11
Pu	2.2	0.26	− 0.4	− 0.024	1.02	0.24	0.14	0.02
Cm	2.2	0.26	− 0.7	− 0.041	0.96	0.11	0.06	0.00
Cf	1.5	0.18	0.0	0.000	0.73	0.22	0.03	0.01
Fm	1.2	0.14	0.4	0.024	0.65	0.30	0.09	0.02

to be different from zero. A rough estimate indicates that β_6- and β_8-values of the order of magnitude of 0.02 should be employed to account for the large b_6- and b_8-values which have been found empirically for the parent nuclei of Pu and Cm.

By means of Coulomb excitation experiments one could (at least in principle) obtain further information on the higher order multipole deformations which have been considered in this section, and also on the possible nuclear pear-shape which will be considered in the following section.

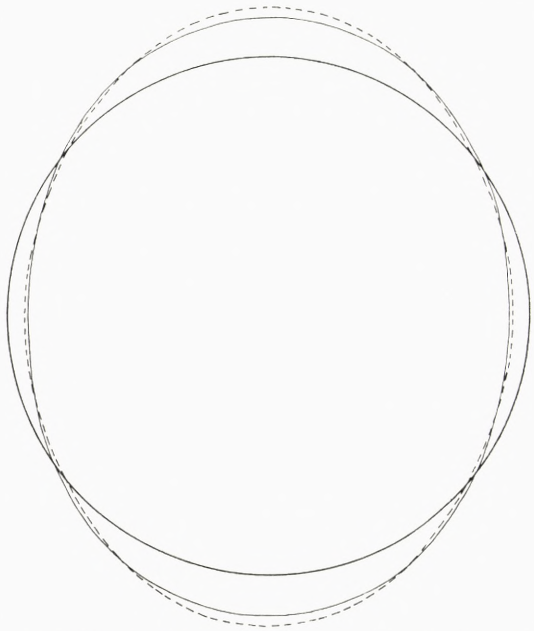

Fig. 3. Two possibilities for the shape of a strongly deformed nucleus.
The circle represents $r = R_0$
The full-drawn curve represents $r = R_0 \{1 + 0.25\, Y_{2,0}\,(\vartheta')\}$
The dotted curve represents $r = R_0 \{1 + 0.25\, Y_{2,0}\,(\vartheta') + 0.05\, Y_{4,0}\,(\vartheta')\}$

c. Odd parity alpha transitions in even-even nuclei.

The even parity alpha groups $\alpha_0, \alpha_2, \ldots$ which we have discussed in the previous section are characteristic of all the strongly deformed even-even alpha emitters. For some isotopes of Th and U (and for Cm^{242}) one has found also odd parity alpha groups [52, 53] which correspond to the nuclear transitions $0+ \rightarrow 1-, 3-, \ldots$ and which will be denoted by $\alpha_1, \alpha_3, \ldots$, respectively. The knowledge available on these odd parity alpha groups and on the corresponding odd parity states in the daughter nuclei is collected in the Tables 8 and 9*. The relatively small excitation energies of the 1−

*Note added in proof: More recent data on the alpha groups populating the 1− states found in some even-even nuclei are given in the forthcoming review article on alpha decay by Perlman and Rasmussen[43a] (see their fig. 15). In addition to improvements of the data in our Tables 8 and 9, they give data on the $0+ \longrightarrow 1-$ transitions of the parent nuclei Ra^{222} and Ra^{224}.

TABLE 8. Data (from STEPHENS et al.[52]) on the 1- states which have been found in some even-even daughter nuclei.

Alpha decaying parent nucleus	Excitation energy of the 1- state in the daughter nucleus (keV)	Hindrance factor of the α-group going to the 1- state in the daughter nucleus	Degree of certainty of the 1- assignment
Th^{226}	242	4.0	certain
Th^{228}	217	10	certain
Th^{230}	253	40	certain
U^{230}	232	11	certain
U^{232}	326	70	almost sure
Cm^{242}	605	1000	somewhat doubtful

states indicate that these states have the same individual particle configuration as the even parity states. There is also other evidence[52] that the odd parity alpha groups leave the daughter nucleus in a rotational band with $K_f = 0$ and that therefore the corresponding alpha particles are formed with $\Omega = 0$. The odd parity states may be due to a soft asymmetric vibration of the type $\beta_3 Y_{3,0}(\vartheta')$ in the shape of the nuclear surface. (See, for instance, the review article on Coulomb excitation by ALDER et al.[1].) A special case of such a vibration is that of a nucleus which is pear-shaped in equilibrium and which may oscillate between the mirror shapes. Because of the rather long period for the vibrational motion it appears possible to consider the deformation parameter β_3 as approximately constant during the penetration of the alpha particle through the interior part of the potential barrier. Since the shape of the nuclear surface is expected to be more important for the alpha decay than the anisotropy of the electrostatic field, the formula (VIII-10) can be used approximately also for the odd

TABLE 9. Empirical b_l-values for the odd parity alpha groups of even-even nuclei. (The b_l-values are taken from Table 2).

Parent nucleus	b_1	b_3
Th^{226}	0.55	
Th^{228}	0.35	0.23
Th^{230}	0.17	
U^{230}	0.33	
U^{232}	0.13	
Cm^{242}	0.035	

parity alpha groups, even though it has been derived on the assumption that the differential equations for the alpha penetration problem do not couple the alpha groups going to different bands of the daughter nucleus. By defining c_l and b_l for the odd parity alpha groups in the same way as for the even parity alpha groups we then see that also the formulae (VIII-11), (VIII-12), and (VIII-13) apply to the odd parity alpha groups. The quantities $k_{l,\,l'}(B)$ which appear in these formulae are different from zero only if l and l' are either both even or both odd*. For odd values (1, 3, 5, 7, . . .) of both l and l' we give the following numerical values of $k_{l,\,l'}(B)$, written in matrix form:

$$
\begin{array}{r c}
 & \begin{array}{ccccc} l=1 & l=3 & l=5 & l=7 & \end{array} \\
k(-3) = & \left\{\begin{array}{ccccc}
0.61 & -0.68 & 0.37 & -0.14 & \dots \\
-0.68 & 1.20 & -0.92 & 0.44 & \dots \\
0.37 & -0.92 & 1.28 & -0.94 & \dots \\
-0.14 & 0.44 & -0.94 & 1.28 & \dots \\
\dots & \dots & \dots & \dots & \dots
\end{array}\right\} \\
k(-2) = & \left\{\begin{array}{ccccc}
0.62 & -0.49 & 0.19 & -0.05 & \dots \\
-0.49 & 0.97 & -0.59 & 0.21 & \dots \\
0.19 & -0.59 & 0.99 & -0.59 & \dots \\
-0.05 & 0.21 & -0.59 & 1.00 & \dots \\
\dots & \dots & \dots & \dots & \dots
\end{array}\right\} \\
k(-1) = & \left\{\begin{array}{ccccc}
0.73 & -0.30 & 0.05 & -0.01 & \dots \\
-0.30 & 0.88 & -0.31 & 0.06 & \dots \\
0.05 & -0.31 & 0.89 & -0.31 & \dots \\
-0.01 & 0.06 & -0.31 & 0.89 & \dots \\
\dots & \dots & \dots & \dots & \dots
\end{array}\right\}
\end{array}
$$

$k(0) =$ the unit matrix, i. e., $k_{l,\,l'}(0) = \delta_{l,\,l'}$

$$
\begin{array}{r c}
k(1) = & \left\{\begin{array}{ccccc}
1.60 & 0.58 & 0.10 & 0.01 & \dots \\
0.58 & 1.51 & 0.53 & 0.10 & \dots \\
0.10 & 0.53 & 1.48 & 0.52 & \dots \\
0.01 & 0.10 & 0.52 & 1.48 & \dots \\
\dots & \dots & \dots & \dots & \dots
\end{array}\right\} \\
k(2) = & \left\{\begin{array}{ccccc}
2.92 & 1.85 & 0.64 & 0.17 & \dots \\
1.85 & 2.91 & 1.70 & 0.58 & \dots \\
0.64 & 1.70 & 2.78 & 1.66 & \dots \\
0.17 & 0.58 & 1.66 & 2.76 & \dots \\
\dots & \dots & \dots & \dots & \dots
\end{array}\right\} \\
k(3) = & \left\{\begin{array}{ccccc}
5.82 & 4.83 & 2.32 & 0.79 & \dots \\
4.83 & 6.42 & 4.59 & 2.14 & \dots \\
2.32 & 4.59 & 6.02 & 4.39 & \dots \\
0.79 & 2.14 & 4.39 & 5.92 & \dots \\
\dots & \dots & \dots & \dots & \dots
\end{array}\right\} \\
 & \begin{array}{ccccc} l=1 & l=3 & l=5 & l=7 & \end{array}
\end{array}
$$

* Therefore the b_l-values for the even parity alpha groups depend on $a_0, a_2, a_4, \dots$, but not on $a_1, a_3, a_5, \dots$, whereas the b_l-values for the odd parity alpha groups depend on both these two sets of a_l-values.

By assuming the alpha wave function on the nuclear surface, $\psi_0(\vartheta')$, to be constant and by using (VI-10), (VI-11), and (VIII-13), we obtain to first approximation

$$b_1 \approx \left|\frac{k_{1,3}(B)\, a_3}{k_{0,0}(B)\, a_0}\right| \approx \left|\frac{k_{1,3}(B)}{k_{0,0}(B)}\right| \varkappa \left|\beta_3\right| \sqrt{\frac{kR_0}{\varkappa}\left(1-\frac{kR_0}{\varkappa}\right)}$$

and

$$\frac{b_3}{b_1} \approx \left|\frac{k_{3,3}(B)}{k_{1,3}(B)}\right|,$$

$|\beta_3|$ being understood as an average value of $|\beta_3|$. Substituting appropriate numerical values in these two formulae, we get

$$|\beta_3| \sim 0.05\, b_1$$

and

$$\frac{b_3}{b_1} \sim 1.5.$$

From the estimate of $|\beta_3|$ and the b_1-values in Table 9 we find that $|\beta_3| \sim 0.01$ to 0.03 for the daughter nuclei corresponding to the parent nuclei of Th and U, and that $|\beta_3| \sim 0.002$ for the daughter nucleus corresponding to the parent nucleus Cm^{242}. The estimate of b_3/b_1 can at present be tested only for the parent nucleus Th^{228} for which $b_3/b_1 = 0.23/0.35 = 0.66$ according to Table 9*. For other parent nuclei for which the alpha group α_1 has been found, the estimate af b_3/b_1 may give information about the possibility of detecting the alpha group α_3.

d. Favoured alpha transitions in odd-A nuclei.

An especially interesting type of alpha decay is the favoured alpha decay which is characterized by the alpha particle being formed from paired nucleons moving in orbitals differing only in the sign of the angular momentum component along the nuclear axis[15]. In this type of alpha decay, the alpha particle is therefore formed with the angular momentum component $\Omega = 0$, and hence the daughter nucleus is left in a band with $K_f = K_i$. The alpha decay of an even-even nucleus is of the favoured type. The same is also true for the alpha decay of an odd-A nucleus if the last odd nucleon moves in the same orbital in the daughter nucleus as in the parent nucleus. All the favoured alpha groups of odd-A nuclei which have been identified have even parity.

Nucleons occupying states which differ only in the sign of the angular momentum component along the nuclear axis interact especially strongly due to the large overlap of their wave functions. Hence it is most probable for an alpha particle to be formed from such paired nucleons, i. e., favoured alpha decay is intrinsically more probable than other types of alpha decay. The actual alpha transition probabilities are, however,

* Note added in proof: Quite recent, unpublished data on odd parity alpha transitions in even-even uclei indicate similar discrepancies between the theoretical and empirical values of b_3/b_1 to those for the arent nucleus Th^{228} (private communication from Dr. MOTTELSON).

strongly affected by the potential barrier, and in many cases they are therefore much smaller for the favoured than for some unfavoured alpha groups.

Because of the large intrinsic probability for favoured alpha decay, the favoured alpha groups of even parity emerging from an odd-A nucleus are recognized by the fact that their F-values (see Chapter VII) are comparatively large, being less than 1, but still of this order of magnitude for small spin changes. Thus, the F-value for the special alpha group of this type which corresponds to the nuclear transition involving no change of spin is found to be about 0.5 on the average (see Table 1). We may express this fact simply by saying that for odd-A nuclei the favoured alpha groups of even parity are intrinsically hindered by a factor of about 2 compared with the corresponding alpha groups for even-even nuclei.

For favoured alpha decay the alpha wave function on the nuclear surface, $\psi_0(\vartheta')$, contains only the angular momentum component $\Omega = 0$ and is therefore axially symmetric with respect to the nuclear axis. This fact implies that the coefficients $a_{l,\Omega}$ defined by (VI-11) are different from zero only for $\Omega = 0$. For favoured alpha groups the formula (VI-13) therefore becomes

$$\left.\begin{aligned} P_{K_f, I_f} &= v_{I_f}\left(\frac{R_0}{G_0(E_{I_f}, R_0)}\right)^2 \sum_l (I_i, l; K_f, 0 \mid I_i, l; I_f, K_f)^2 \\ &\times \left|\sum_{l'} k_{l,l'}(B)\, a_{l'}\right|^2 \exp\left\{-\frac{2\,l(l+1)}{\varkappa}\sqrt{\frac{\varkappa}{kR_0}-1}\right\}. \end{aligned}\right\} \qquad \text{(VIII-17)}$$

We define the reduced transition probability c_{K_f, I_f} for the favoured alpha transition $(K_i, I_i) \to (K_f, I_f)$ as the quotient of the probability for this favoured alpha transition and the probability for the favoured alpha transition which involves no change of spin and parity, where the latter probability is corrected such that it corresponds to an alpha transition energy equal to that for the transition $(K_i, I_i) \to (K_f, I_f)$. From this definition and (VIII-17), and the fact that the essential energy dependence in this expression for P_{K_f, I_f} enters through $G_0(E_{I_f}, R_0)$, we get

$$c_{K_f, I_f} = \frac{\sum_l c_l' (I_i, l; K_f, 0 \mid I_i, l; I_f, K_f)^2}{\sum_l c_l' (I_i, l; K_f, 0 \mid I_i, l; I_i, K_f)^2}, \qquad \text{(VIII-18)}$$

where

$$c_l' = \left|\sum_{l'} k_{l,l'}(B)\, a_{l'}\right|^2 \exp\left\{-\frac{2\,l(l+1)}{\varkappa}\sqrt{\frac{\varkappa}{kR_0}-1}\right\}. \qquad \text{(VIII-19)}$$

In the numerator of (VIII-18) the sum over l is restricted either to only even l-values or to only odd l-values depending on whether the parities of the parent and daughter nuclei are the same or the opposite (cf. the last few lines in this section). In the denominator of (VIII-18) the sum over l is always restricted to only even l-values.

The favoured alpha decay of odd-A nuclei must in many respects be very similar to that of even-even nuclei. It would therefore be natural to assume that, for favoured alpha decay of an odd-A nucleus, the alpha wave function on the nuclear surface, $\psi_0(\vartheta')$, could be calculated by interpolating the corresponding functions $\psi_0(\vartheta')$ for neighbouring even-even nuclei to the Z-value of the odd-A nucleus considered. However, according to the formulae (VIII-10) and (VIII-17), this assumption

TABLE 10. Data on those favoured alpha decays for which one can test the formula which relates the relative intensities of the favoured alpha groups of an odd-A nucleus to the relative intensities of the alpha groups of neighbouring even-even nuclei.

Parent nucleus	$K_f = K_i = I_i$	I_f	E_α (MeV)	c_{K_f, I_f} emp.	c_l				c_{K_f, I_f} theor.
					$l = 0$	$l = 2$	$l = 4$	$l = 6$	
		3/2	4.727	1					1
$_{91}Pa^{231}$	3/2	5/2	4.671	0.32	1	0.87	0.091		0.38
		7/2	4.593 (calc.)						0.23
		5/2	4.8157	1					1
$_{92}U^{233*}$	5/2	7/2	4.7732	0.34	1	0.83	0.071		0.31
		9/2	4.7174	0.093					0.13
		11/2	4.650 (calc.)						0.018
		1/2	5.150	1					1
$_{94}Pu^{239}$	1/2	3/2	5.137	0.33	1	0.65	0.011	0.0026	0.26
		5/2	5.099	0.35					0.39
		7/2	5.070 (calc.)						0.0049
		5/2	5.4820	1					1
		7/2	5.4391	0.31					0.24
$_{95}Am^{241}$	5/2	9/2	5.3860	0.083	1	0.59	0.0040	0.0031	0.083
		11/2	5.3210	0.0018					0.0016
		13/2	5.241	0.00076					0.0013
		5/2	5.267	1					1
$_{95}Am^{243}$	5/2	7/2	5.224	0.25	1	0.59	0.0040	0.0031	0.24
		9/2	5.169	0.052					0.083
		11/2	5.102 (calc.)						0.0016
		5/2	5.777	1					1
$_{96}Cm^{243}$	5/2	7/2	5.732	0.29	1	0.57	0.0017	0.0033	0.23
		9/2	5.674 (calc.)						0.080
		7/2	6.633	1					1
$_{99}E^{253}$	7/2	9/2	6.592	0.13	1	0.29	0.014	0.0022	0.11
		11/2	6.545	0.050					0.033
		13/2							0.0034

Note added in proof: Recently, GOLDIN et al.[31a] have investigated the decay af U^{233} and observed two new alpha groups corresponding to transitions to the states of the daughter nucleus with spins 11/2 and 13/2. The intensity values, which are given with large uncertainties, are only in qualitative agreement with the formula (VIII-18).

does not account for the previously mentioned hindrance by a factor of about 2 for the favoured alpha groups of even parity in odd-A nuclei. Therefore we make the somewhat more complicated assumption that, for favoured alpha decay of an odd-A nucleus, the function $\psi_0(\vartheta')$ is equal to the function which one obtains by first interpolating the corresponding functions $\psi_0(\vartheta')$ for neighbouring even-even nuclei to the odd-A nucleus considered and then multiplying the resulting function by an angular-independent factor having an absolute value of about 0.7 $(=\sqrt{0.5})$. According to (VI-10), (VI-11), (VIII-11), and (VIII-19), the quantities c_l' for our odd-A nucleus are then proportional to the reduced alpha transition probabilities c_l for even-even nuclei, interpolated to the odd-A nucleus considered. In the formula (VIII-18) we may therefore simply replace the c_l'-values by these interpolated reduced transition probabilities c_l. In this way, we get an intensity formula which makes it possible to calculate the reduced transition probabilities for the favoured alpha groups of odd-A nuclei on the basis of the empirically determined reduced transition probabilities c_l for the alpha groups of even-even nuclei. In a slightly different form, this formula was first given in a paper by Bohr, Fröman, and Mottelson[15]. There, this intensity formula was tested by comparing, for each possible odd-A parent nucleus, the relative values of the empirical transition probabilities for the favoured alpha groups with the relative values of the corresponding theoretical transition probabilities. A more sensitive way of testing the formula is, however, to compare the empirical values of c_{K_f, I_f} for the favoured alpha groups of the odd-A nuclei (obtained from Table 1) with the corresponding theoretical values. This comparison is given in Table 10 which lists those favoured alpha groups of odd-A nuclei for which sufficient empirical data are available. The values of $K_f\,(=K_i)$ and I_i are taken from the Tables III and IV in the previously mentioned paper[15], except for the parent nucleus E^{253} for which the suggested identification of the favoured alpha groups is new*. For this nucleus we have assumed that $K_f = 7/2$. The values of the alpha particle energies E_α and the empirical values of the reduced transition probabilities c_{K_f, I_f} are taken from Table 1 in the present paper. The c_l-values are obtained by interpolating between the empirical c_l-values for the even-even nuclei (see Table 2). The theoretical values of c_{K_f, I_f} have been calculated from (VIII-18) by neglecting the terms corresponding to $l > 6$ (and in most cases also the term corresponding to $l = 6$). When the empirical and the theoretical values of c_{K_f, I_f} in Table 10 are compared, one finds an agreement which is as good as can be expected considering the approximations involved in the theoretical formula and the uncertainties in the experimental data.

Although no favoured alpha groups of odd parity have so far been identified for odd-A nuclei, the existence of odd parity alpha groups for some of the heavy even-even nuclei suggests the possibility for the existence of favoured alpha groups of odd parity for neighbouring odd-A nuclei. In the case of such an odd-A nucleus,

* Note added in proof: The identification of the favoured alpha groups of E^{253} is also given in the forthcoming review article on alpha decay by Perlman and Rasmussen[43a] (their table XVI). Furthermore, these authors suggest an identification for the favoured alpha groups of Pu^{241}.

the two bands in which the favoured alpha groups of even and odd parity, respectively, leave the daughter nucleus are expected to have similar level structure, but opposite parity, and to be displaced with respect to each other by roughly the same amount as the even and odd parity bands in neighbouring even-even nuclei, i. e., by a few hundred keV. The possibility for detecting the odd parity alpha groups here suggested can be estimated on the basis of the intensity formula (VIII-18) and the empirical c_l-values for the even-even nuclei.

e. Angular distributions of the alpha groups from polarized nuclei.

The expression (VI-14) for the angular distributions of the alpha groups emerging from polarized nuclei can be evaluated by means of the formula for the expansion of a product of two spherical harmonics in terms of spherical harmonics and by means of a sum rule[11a] which expresses sums (over magnetic quantum numbers) of products of three Clebsch-Gordan coefficients in terms of Racah coefficients. Thus, if the coefficients $a_{l,\Omega}$ are assumed to be real, the expression (VI-14) becomes

$$\left.\begin{aligned} &v_{I_f}\left(\frac{R_0}{G_0(E_{I_f}, R_0)}\right)^2 \frac{1}{4\pi} \sum_L (-1)^{I_f-I_i} \sqrt{\frac{2L+1}{2I_i+1}} (I_i, L; M_i, 0 \,|\, I_i, L; I_i, M_i) \\ &\times \left\{ \sum_l \sum_{l'} (-1)^{\frac{1}{2}(L-l+l')} \cos \frac{l(l+1)-l'(l'+1)}{\varkappa} Z(l, I_i, l', I_i; I_f, L) F_l F_{l'} \right\} P_L(\cos\vartheta), \end{aligned}\right\} \quad \text{(VIII-20)}$$

where

$$\left.\begin{aligned} F_l = \exp&\left\{-\frac{l(l+1)}{\varkappa}\sqrt{\frac{\varkappa}{kR_0}-1}\right\} \\ \times \sum_\Omega (-1)^\Omega &(I_i, l; K_f+\Omega, -\Omega \,|\, I_i, l; I_f, K_f) \sum_{l'} k_{l,l'}^{\Omega}(B)\, a_{l',\Omega} \end{aligned}\right\} \quad \text{(VIII-21)}$$

and the coefficients Z are related to the Racah coefficients W according to the formula[11a]

$$\begin{aligned} &Z(a, b, c, d; e, f) \\ &= (-1)^{\frac{1}{2}(f-a+c)} \sqrt{(2a+1)(2b+1)(2c+1)(2d+1)}\, W(a, b, c, d; e, f)(a, c; 0, 0 \,|\, a, c; f, 0). \end{aligned}$$

The expression (VIII-20) is to be averaged with respect to the initial distribution of M_i-values. This averaging affects only the Clebsch-Gordan coefficient which appears in (VIII-20). For $L = 0$, this Clebsch-Gordan coefficient is always equal to 1, and for $L = 2$ its average value with respect to the initial distribution of M_i-values is

$$(I_i, 2; M_i, 0 \,|\, I_i, 2; I_i, M_i)_{Av} = \frac{6\, I_i^2}{\sqrt{(2I_i-1)\, 2I_i\, (2I_i+2)(2I_i+3)}} \Delta, \quad \text{(VIII-22)}$$

where Δ is defined by

$$\Delta = \frac{1}{I_i^2} \left\langle M_i^2 - \tfrac{1}{3} I_i (I_i+1) \right\rangle_{Av} \quad \text{(VIII-23)}$$

and is a measure of the polarization of the parent nuclei, being equal to zero for non-polarized parent nuclei. Considering only even L-values and using (VIII-22), we can write the above mentioned average value of the expression (VIII-20) as follows:

$$v_{I_f}\left(\frac{R_0}{G_0(E_{I_f}, R_0)}\right)^2 \frac{\sum_l F_l^2}{4\pi}\left\{1 + C_{I_i, I_f}(B)\,\Delta P_2(\cos\vartheta) + \ldots\right\}, \tag{VIII-24}$$

where

$$\left.\begin{aligned} C_{I_i, I_f}(B) &= \frac{(-1)^{I_f - I_i}\, 6\sqrt{5}\, I_i^2}{\sqrt{(2I_i-1)\,2I_i\,(2I_i+1)(2I_i+2)(2I_i+3)}} \\ &\times \frac{\sum_l \sum_{l'} (-1)^{\frac{1}{2}(2-l+l')} \cos\frac{l(l+1)-l'(l'+1)}{\varkappa}\, Z(l, I_i, l', I_i; I_f, 2)\, F_l F_{l'}}{\sum_l F_l^2} \end{aligned}\right\} \tag{VIII-25}$$

The case of favoured alpha decay of parent nuclei of spin 5/2 is especially interesting for the interpretation of the experimental results available on polarized nuclei[22, 46, 46a, 47]. Assuming the function $\psi_1(\vartheta')$ to be independent of ϑ' (i.e., all the coefficients a_l, except a_0, to be zero), we obtain the numerical values of the coefficient $C_{\frac{5}{2},\frac{5}{2}}$ listed in Table 11. The quadrupole deformations found for the very heavy nuclei

Table 11.
Values of $C_{\frac{5}{2},\frac{5}{2}}$ (B) for the case of favoured alpha decay corresponding to $\varkappa = 50$, $kR_0/\varkappa = 0.2$, and $\psi_1(\vartheta')$ being independent of ϑ'.

B	$C_{\frac{5}{2},\frac{5}{2}}(B)$
−2	−1.26
−1	−0.83
0	0
1	1.15
2	2.07

correspond to prolate nuclear shapes, i.e., to positive B-values. For such B-values, the coefficient $C_{\frac{5}{2},\frac{5}{2}}$ is positive, while the experimental results appear to indicate an alpha distribution peaked perpendicular to the direction of polarization[22, 46, 46a, 47]. We abstain from giving values for $C_{\frac{5}{2},\frac{7}{2}}$.

Appendix A.

Formulae and Graphs for the Gamow Wave Functions.

In this appendix, we collect some formulae* for GAMOW's well-known *WKB*-solution of the differential equation describing the penetration of an alpha particle of given angular momentum l through a spherically symmetric Coulomb barrier. For $l = 0$, we also give graphical illustrations of some properties of this solution.

Consider the differential equation

$$\left\{-\frac{\hbar^2}{2\,m}\frac{d^2}{dr^2} + U_l(r) - E\right\} G_l(E, r) = 0, \tag{A-1}$$

where

$$U_l(r) = \frac{2\,(Z-2)\,\varepsilon^2}{r} + \frac{\hbar^2\, l(l+1)}{2\,m\,r^2}.$$

The Gamow wave function $G_l(E, r)$ is a solution of (A-1) representing an outgoing wave for large values of r. Such a solution is unique except for an arbitrary constant factor, and by choosing this factor conveniently, we get

$$G_l(E, r) = \left\{\begin{array}{ll} \left(\dfrac{U_l - E}{E}\right)^{-\frac{1}{4}} \exp\left\{k\displaystyle\int_r \left(\dfrac{U_l - E}{E}\right)^{\frac{1}{2}} dr\right\} & \text{when } U_l > E \\ \left(\dfrac{E - U_l}{E}\right)^{-\frac{1}{4}} \exp\left\{i\dfrac{\pi}{4} + ik\displaystyle\int^r \left(\dfrac{E - U_l}{E}\right)^{\frac{1}{2}} dr\right\} & \text{when } U_l < E \end{array}\right\} \tag{A-2}$$

according to the *WKB*-approximation, which in general is considered to be valid if the condition

$$\left|\frac{1}{k}\frac{d}{dr}\left(\left|\frac{U_l - E}{E}\right|^{-\frac{1}{2}}\right)\right| << 1 \tag{A-3}$$

is fulfilled. The two integration limits in (A-2), which are not written out, both correspond to the point where U_l, which is an abbreviation of $U_l(r)$, is equal to E. The quantity k which appears in (A-2) and (A-3) is defined by

$$k = \frac{1}{\hbar}\sqrt{2\,m\,E},$$

and represents the magnitude of the wave vector for large values of r. The corresponding velocity is

$$v = \sqrt{\frac{2\,E}{m}}.$$

For the evaluation of (A-2) it is convenient to introduce the dimensionless quantity

* Many of these formulae are already known and can be found, e. g., in the book by GAMOW and CRITCHFIELD[28].

$$\varkappa = \frac{4\,(Z-2)\,\varepsilon^2}{\hbar v} \tag{A-4}$$

which is well-known in atomic collision problems. In our case, $\varkappa$ is of the order of magnitude of 50. The fact that $\varkappa >> 1$ is a criterion that a classical orbital picture is possible outside the potential barrier[17].

For the quantity $(U_l - E)/E$ which appears in (A-2) and (A-3), we now easily find that

$$\frac{U_l - E}{E} = \left(\frac{\varkappa}{kr} - 1\right)\left\{1 + \frac{l\,(l+1)}{\varkappa^2}\,\frac{1}{\dfrac{kr}{\varkappa}\left(1 - \dfrac{kr}{\varkappa}\right)}\right\}. \tag{A-5}$$

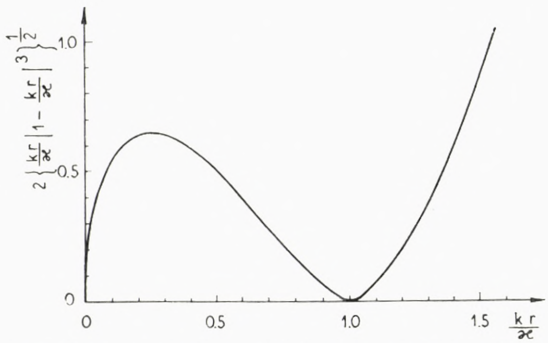

Fig. 4. Graphical representation of the expression on the left-hand side in the condition (A-6) for the validity of the WKB-approximation.

In the condition (A-3) it is for our purpose a sufficiently good approximation to replace $U_l\,(r)$ by $U_0\,(r)$. According to (A-5), this is the same as replacing $(U_l - E)/E$ by $\varkappa/kr - 1$. In this way the condition (A-3) becomes

$$2\left(\frac{kr}{\varkappa}\left|1 - \frac{kr}{\varkappa}\right|^3\right)^{\frac{1}{2}} >> \frac{1}{\varkappa}. \tag{A-6}$$

A graphical representation of the expression on the left-hand side of this inequality is shown in Fig. 4, from which it is seen that the condition (A-6) is fulfilled except for a small region around the classical turning point.

In the following we shall treat the two cases $kr < \varkappa$ and $kr > \varkappa$ separately.

a. *The case* $kr < \varkappa$, *i. e.*, $U_0\,(r) > E$.

Instead of r we introduce the variable α which we define by

$$\cos^2\alpha = \frac{kr}{\varkappa} = \frac{E}{U_0(r)} \qquad \left(0 < \alpha < \frac{\pi}{2}\right). \tag{A-7 a}$$

We note the two formulae

$$\frac{\partial (\varkappa \cos \alpha)}{\partial E} = 0 \tag{A-8a}$$

and

$$\frac{\partial \alpha}{\partial E} = -\frac{\cot \alpha}{2E} \tag{A-9a}$$

for later use.

The condition (A-6) for the validity of the *WKB*-approximation becomes

$$2 \sin^3 \alpha \cos \alpha >> \frac{1}{\varkappa}. \tag{A-10a}$$

From (A-2) (for $U_l > E$), (A-5), (A-7a), and (A-10a) we get the well-known approximate formula

$$G_l(E, r) = (\cot \alpha)^{\frac{1}{2}} \exp\left\{\varkappa(\alpha - \sin \alpha \cos \alpha) + \frac{l(l+1)}{\varkappa} \operatorname{tg} \alpha\right\} \tag{A-11a}$$

which is valid if the centrifugal barrier is small compared to the Coulomb barrier.

We shall now calculate $G_l(E + \Delta E, r)/G_l(E, r)$. Obviously,

$$\ln \frac{G_l(E + \Delta E, r)}{G_l(E, r)} \approx \left\{\frac{\partial}{\partial E} \ln G_l(E, r)\right\} \Delta E.$$

Using this formula, together with (A-8a), (A-9a), and (A-11a), we get

$$\frac{G_l(E + \Delta E, r)}{G_l(E, r)} \approx \exp\left\{-\left[\tfrac{1}{2}\varkappa(\alpha + \sin \alpha \cos \alpha) + \frac{l(l+1)}{2\varkappa} \cot \alpha\right]\frac{\Delta E}{E}\right\} \tag{A-12a}$$

if the condition (A-10a) is also taken into account.

The formulae (A-11a) and (A-12a) are valid only if it is assumed that the centrifugal potential is small compared to the Coulomb potential. If this condition is not introduced, one finds, by a straight-forward evaluation of the integral appearing in the *WKB*-approximation formula (A-2) for $U_l > E$, that[23]

$$G_l(E, r) = \left[\frac{l(l+1)}{\varkappa^2}\left(\frac{\varkappa}{kr}\right)^2 + \frac{\varkappa}{kr} - 1\right]^{-\frac{1}{4}} \exp\left\{\varkappa F\left(\frac{l(l+1)}{\varkappa^2}, \frac{kr}{\varkappa}\right)\right\}, \tag{A-13a}$$

where

$$F(a, x) = \sqrt{a} \ln \frac{x + 2a + \sqrt{4a(a + x - x^2)}}{x\sqrt{1 + 4a}} + \tfrac{1}{2} \arccos \frac{2x - 1}{\sqrt{1 + 4a}} - \sqrt{a + x - x^2} \tag{A-14a}$$

and arccos shall be chosen to have a value between 0 and π. If we suppose $l(l+1)/\varkappa^2$ to be small and expand $F(l(l+1)/\varkappa^2, kr/\varkappa)$ in terms of powers of $l(l+1)/\varkappa^2$, keeping only terms that are linear in $l(l+1)/\varkappa^2$, we find from (A-13a) and (A-14a) the approximate formula

$$G_l(E, r) = G_0(E, r) \exp\left\{\frac{l(l+1)}{\varkappa}\sqrt{\frac{\varkappa}{kr} - 1}\right\} \tag{A-15a}$$

which gives the l-dependence already found in (A-11a).

TABLE 12. For the case $\varkappa = 50$ and $kR_0 = 0.2\,\varkappa$, values of the relative penetration probabilities for different angular momenta l are listed. They are given in two different degrees of approximation. The last column corresponds to column 3 in Table 13.

l	$\{G_0(E, R_0)/G_l(E, R_0)\}^2$ according to (A-13a) and (A-14a)	according to (A-15a)	l	$\{G_0(E, R_0)/G_l(E, R_0)\}^2$ according to (A-13a) and (A-14a)	according to (A-15a)
0	1	1	8	$0.386\cdot10^{-2}$	$0.315\cdot10^{-2}$
1	0.854	0.852	9	$0.995\cdot10^{-3}$	$0.747\cdot10^{-3}$
2	0.624	0.619	10	$0.224\cdot10^{-3}$	$0.151\cdot10^{-3}$
3	0.390	0.383	11	$0.440\cdot10^{-4}$	$0.259\cdot10^{-4}$
4	0.209	0.202	12	$0.762\cdot10^{-5}$	$0.380\cdot10^{-5}$
5	$0.961\cdot10^{-1}$	$0.907\cdot10^{-1}$	13	$0.116\cdot10^{-5}$	$0.475\cdot10^{-6}$
6	$0.381\cdot10^{-1}$	$0.347\cdot10^{-1}$	14	$0.158\cdot10^{-6}$	$0.506\cdot10^{-7}$
7	$0.130\cdot10^{-1}$	$0.113\cdot10^{-1}$	15	$0.190\cdot10^{-7}$	$0.459\cdot10^{-8}$

Using formulae equivalent to (A-13a) and (A-14a), DEVANEY[23] has calculated the relative penetration probabilities $\{G_0(E, R_0)/G_l(E, R_0)\}^2$ for alpha particles of different angular momenta l in the case that $Z-2 = 86$, $R_0 = 9.87\cdot10^{-13}$ cm, and $E = 4.88$ MeV (i. e., $\varkappa = 49.1$ and $kR_0/\varkappa = 0.195$). His numerical results (table I on p. 591) which are also listed in BLATT and WEISSKOPF[12] (table 3.1 on p. 577) appear, however, to be considerably too low for large values of l. It is probable that DEVANEY has calculated $\{G_0(E, R_0)/G_l(E, R_0)\}^4$ instead of $\{G_0(E, R_0)/G_l(E, R_0)\}^2$. For the parameter values $\varkappa = 50$ and $kR_0/\varkappa = 0.2$, which differ only slightly from those

TABLE 13. Relative penetration probabilities for different angular momenta l, calculated according to the simple approximate expression $\exp\{-cl(l+1)\}$. The values chosen for the parameter c, which is related to the atomic parameters according to the formula (A-16a), cover the range of practical interest.

l	$\exp\{-cl(l+1)\}$ for $c = 0.07$	for $c = 0.08$	for $c = 0.09$	for $c = 0.10$
0	1	1	1	1
1	0.870	0.852	0.835	0.819
2	0.657	0.619	0.583	0.549
3	0.432	0.383	0.340	0.301
4	0.247	0.202	0.165	0.135
5	0.123	$0.907\cdot10^{-1}$	$0.672\cdot10^{-1}$	$0.498\cdot10^{-1}$
6	$0.528\cdot10^{-1}$	$0.347\cdot10^{-1}$	$0.228\cdot10^{-1}$	$0.150\cdot10^{-1}$
7	$0.198\cdot10^{-1}$	$0.113\cdot10^{-1}$	$0.647\cdot10^{-2}$	$0.370\cdot10^{-2}$
8	$0.647\cdot10^{-2}$	$0.315\cdot10^{-2}$	$0.153\cdot10^{-2}$	$0.747\cdot10^{-3}$
9	$0.183\cdot10^{-2}$	$0.747\cdot10^{-3}$	$0.304\cdot10^{-3}$	$0.123\cdot10^{-3}$
10	$0.453\cdot10^{-3}$	$0.151\cdot10^{-3}$	$0.502\cdot10^{-4}$	$0.167\cdot10^{-4}$

corresponding to the case treated numerically by DEVANEY, we give in Table 12 the values of $\{G_0(E, R_0)/G_l(E, R_0)\}^2$ according to (A-13 a) together with (A-14 a), and (A-15 a).

According to the simple approximate formula (A-15 a) the relative penetration probabilities for different angular momenta l are $\exp\{-cl(l+1)\}$, where

$$c = \frac{2}{\varkappa}\sqrt{\frac{\varkappa}{kR_0} - 1} = \frac{2}{\varkappa}\sqrt{\frac{2(Z-2)\varepsilon^2}{E R_0} - 1} \tag{A-16 a}$$

and R_0 is the radius of the spherical nucleus. In Table 13 numerical values of $\exp\{-cl(l+1)\}$ are listed for the range of c-values which is of interest for experimental interpretations.

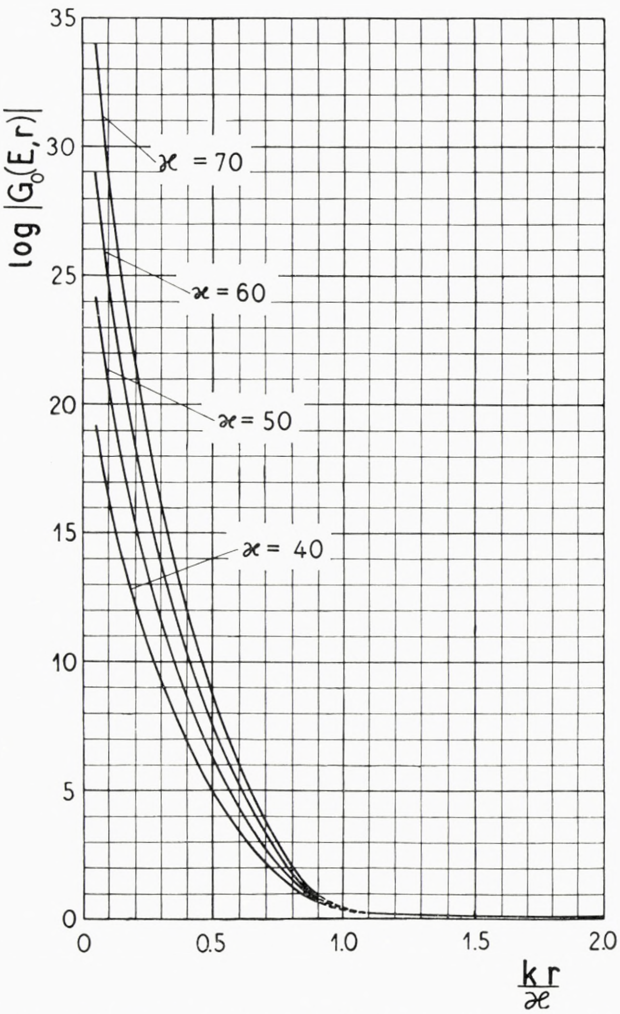

Fig. 5. Graphical representation of the 10-logarithm of $|G_0(E, r)|$.

b. *The case $kr > \varkappa$, i. e., $U_0(r) < E$.*

Instead of r we introduce the variable A, which we define by

$$\cosh^2 A = \frac{kr}{\varkappa} = \frac{E}{U_0(r)} \qquad (A > 0). \tag{A-7b}$$

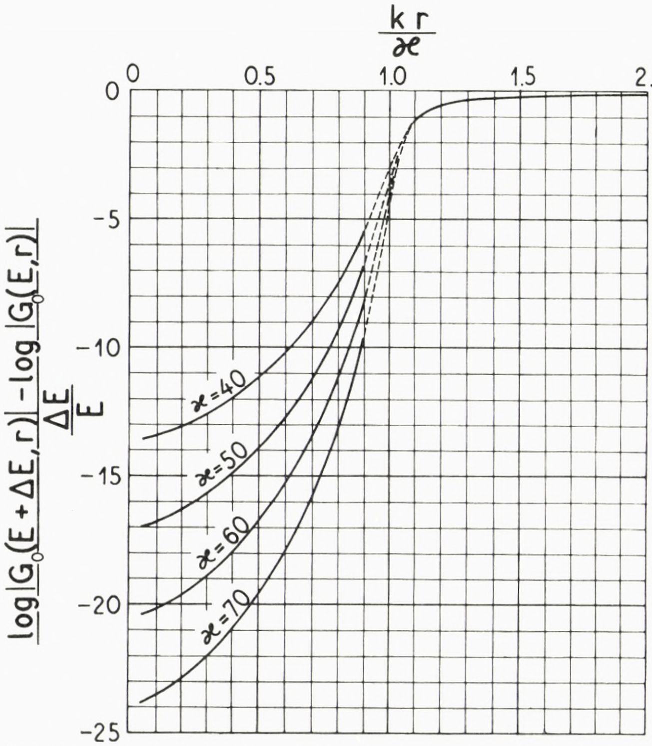

Fig. 6. The increase of the 10-logarithm of $|G_0(E, r)|$, as E is increased by ΔE but r is fixed, is obtained by multiplying the appropriate ordinate in this graph by $\Delta E/E$.

We note the two formulae

$$\frac{\partial(\varkappa \cosh A)}{\partial E} = 0 \tag{A-8b}$$

and

$$\frac{\partial A}{\partial E} = \frac{\coth A}{2E} \tag{A-9b}$$

to be used later.

The condition (A-6) for the validity of the *WKB*-approximation becomes

$$2 \sinh^3 A \cosh A >> \frac{1}{\varkappa}. \tag{A-10b}$$

From (A-2) (for $U_l < E$), (A-5), and (A-7b) we get the approximate formula

$$\left.\begin{aligned} G_l(E,r) &= (\coth A)^{\frac{1}{2}} \exp\left\{\frac{l(l+1)}{4\varkappa^2}\frac{1}{\sinh^2 A\cosh^2 A}\right.\\ &\left.+ i\left[\frac{\pi}{4} + \varkappa(\sinh A\cosh A - A) - \frac{l(l+1)}{\varkappa}\operatorname{tgh} A\right]\right\}. \end{aligned}\right\} \tag{A-11 b}$$

Now, we calculate $|G_l(E+\Delta E, r)/G_l(E,r)|$. Obviously,

$$ln\left|\frac{G_l(E+\Delta E,r)}{G_l(E,r)}\right| \approx \left\{\frac{\partial}{\partial E}\, ln\,|G_l(E,r)|\right\}\Delta E.$$

Using this formula, together with (A-8b), (A-9b), and (A-11b), we get

$$\left|\frac{G_l(E+\Delta E,r)}{G_l(E,r)}\right| \approx \exp\left\{-\left[\frac{1}{4\sinh^2 A} + \frac{l(l+1)}{4\varkappa^2}\frac{1}{\sinh^4 A}\right]\frac{\Delta E}{E}\right\}. \tag{A-12 b}$$

From the formulae in this appendix, $\log|G_0(E,r)|$ and $\dfrac{\log|G_0(E+\Delta E,r)/G_0(E,r)|}{\Delta E/E}$ can easily be found as functions of $kr/\varkappa$ and $\varkappa$. Figs. 5 and 6 give graphical representations of these functions. In connection with these figures, it should be remarked that, in the approximation considered here, $G_0(E,r)$ is real and positive for $kr < \varkappa$, whereas it is complex for $kr > \varkappa$. The same is obviously true for the function $G_0(E+\Delta E, r)/G_0(E,r)$.

Appendix B.

Penetration of an Alpha Particle through a Static Non-central Field.

The wave equation for an alpha particle penetrating a three-dimensional potential barrier, which is fixed in space, is

$$\Delta\psi = K^2\psi, \tag{B-1}$$

where

$$K = K(\mathbf{r}) = \sqrt{\frac{2m}{\hbar^2}(V-E)} = k\sqrt{\frac{V-E}{E}},$$

V denoting the potential, m the mass of the alpha particle, E its kinetic energy after it has escaped completely from the potential barrier, and k the magnitude of the corresponding wave vector. For the construction of an approximate solution of (B-1), Christy has recently developed a three-dimensional *WKB*-method*. By means of such a solution, he can approximately express the alpha wave function in an arbitrary point P inside the potential barrier in terms of the alpha wave function on the nuclear surface, except for a constant factor. In this appendix, we combine Christy's ideas with a method similar to that used by Kirchhoff for the rigorous

* Private communication from Professor R. F. Christy to Professor A. Bohr. See also reference 21.

justification of Huyghens' principle in optics* (see BORN's monograph on optics[18].) The result of these considerations is an approximate integral representation of the alpha wave function inside the potential barrier in terms of the alpha wave function on the nuclear surface, which is closely related to the result of CHRISTY although more rigorously justified. The kernel $H(P, P')$ appearing in our integral representation corresponds, in KIRCHHOFF's formula for light waves in free space, to the spherical wave $\exp\{ik|\boldsymbol{r}_{P'}-\boldsymbol{r}_P|\}/|\boldsymbol{r}_{P'}-\boldsymbol{r}_P|$. Considered as a function of the point P', the kernel $H(P, P')$ is a solution of (B-1) with a singularity behaving as $1/|\boldsymbol{r}_{P'}-\boldsymbol{r}_P|$ as $\boldsymbol{r}_{P'}\to\boldsymbol{r}_P$. We first consider the construction of this kernel by means of the second order approximation of the three-dimensional *WKB*-method, then we derive the previously mentioned integral representation for the alpha wave function. To investigate the consistency of the resulting formula we apply it to the case of a spherically symmetric potential barrier and find that the well-known *WKB*-approximation solution of this problem is very nearly reproduced. Finally, we use our integral representation to obtain a simple approximate solution of the problem concerning the penetration of an alpha particle through an anisotropic potential barrier.

By substituting

$$\psi = e^{-S} \tag{B-2}$$

into (B-1), we get

$$|\mathrm{grad}\, S|^2 - \Delta S = K^2 .$$

In analogy to the procedure in the one-dimensional *WKB*-method, we here neglect the term ΔS and get

$$|\mathrm{grad}\, S| = K. \tag{B-3}$$

This equation is similar to the eikonal equation in geometrical optics (with S/k corresponding to the eikonal) or to the Hamilton-Jacobi equation in classical mechanics (with $\hbar S$ corresponding to HAMILTON's characteristic function). From (B-3) we get

$$\mathrm{grad}\, S = K(\boldsymbol{r})\,\boldsymbol{e}_s , \tag{B-4}$$

where $\boldsymbol{e}_s$ is a unit vector field; its determination will be discussed later. For an arbitrary choice of the vector field $\boldsymbol{e}_s$ there exists in general no solution S of (B-4). The necessary and sufficient condition for the existence of such a solution is that $\mathrm{curl}\,(K\boldsymbol{e}_s) = 0$. It is easily seen that, if there exists a solution S of (B-4), the line integral $\int K\,ds$, taken along a curve which is everywhere tangent to $\boldsymbol{e}_s$ and which terminates on any two surfaces of the type S = constant, has the same value for all such curves between these two surfaces, and it is also seen that, for any two points P and P' on any such curve, this curve is the path between P and P' which minimizes the integral $\int_P^{P'} K\,ds$.

* An extension of Huyghens' principle in optics to the case that the index of refraction is not constant but varies slowly in space can also be obtained along similar lines as those which are described in this appendix.

To construct a special solution of (B-3), we determine the path which joins two points P and P' and which has the extremal property

$$\int_P^{P'} K\, ds = \text{minimum}, \tag{B-5}$$

s being the arc length measured in the direction from P to P'. The unit vector which is tangent to this curve and which points in the direction of increasing arc length s is denoted by $\boldsymbol{e}_s$. The requirement (B-5) is analogous to Fermat's principle in geometrical optics, and the path which one obtains from it is analogous to a light ray. Therefore this path will be called the ray between the points P and P'. From the general formula

$$\delta \int_P^{P'} K\, ds = K\boldsymbol{e}_s \cdot \delta \boldsymbol{r} \Big|_P^{P'} + \int_P^{P'} \left\{ \text{grad}\, K - \frac{d}{ds}(K\boldsymbol{e}_s) \right\} \cdot \delta \boldsymbol{r}\; ds \tag{B-6}$$

it follows that the differential equation for a ray is

$$\text{grad}\, K - \frac{d}{ds}(K\boldsymbol{e}_s) = 0, \tag{B-7}$$

for the first term in (B-6) vanishes for fixed P and P'. By keeping P fixed, but letting P' take all possible positions, we can cover the space by rays which emanate from P. From (B-6) and (B-7) it is then clear that all these rays are cut orthogonally by any surface such that the points P' on it fulfill

$$\int_P^{P'} K\, ds = \text{constant},$$

where the integration is performed along the ray from P to P'. From a simple geometrical consideration it then follows that the condition curl $(K\boldsymbol{e}_s) = 0$ is fulfilled. We now define

$$S = \int_P^{P'} K\, ds, \tag{B-8}$$

where the path of integration is the ray between the given point P and a variable point P'. This function S is obviously a solution of (B-4) and therefore also of (B-3), if $\boldsymbol{e}_s$ on the right-hand side of (B-4) is defined as above by means of the paths obtained from the extremal requirement (B-5). Hence e^{-S}, with S defined by (B-8), is a first order *WKB*-approximation solution of the Schrödinger equation (B-1). For the validity of this approximate solution it is sufficient that simultaneously

$$\left| \text{grad}\, \frac{1}{K} \right| << 1$$

$$\left| \frac{\text{div}\, \boldsymbol{e}_s}{K} \right| << 1.$$

The first of these two inequalities is identical to the well-known condition for the validity of the first order *WKB*-method in one dimension, whereas the second one, which, in the case of a spherically symmetric potential barrier is certainly fulfilled if $r\,K(r)$ is large compared to 1 and if furthermore the point P' does not lie too close to P, appears only for the *WKB*-method in three dimensions.

Next, we derive the second approximation of the solution of (B-1) which we have found above in first approximation. To this purpose, we write

$$\psi = u\,e^{-S}, \tag{B-9}$$

S being still defined according to (B-8). By substituting (B-9) into (B-1), and using (B-4) or (B-8), we get

$$u \operatorname{div}(K\boldsymbol{e}_s) + 2\,K\boldsymbol{e}_s\cdot\operatorname{grad} u - \Delta u = 0\,.$$

In analogy to the procedure in the one-dimensional *WKB*-method we neglect the term Δu and get

$$\boldsymbol{e}_s\cdot\operatorname{grad} ln\left(u K^{\frac{1}{2}}\right) = -\tfrac{1}{2}\operatorname{div}\boldsymbol{e}_s\,,$$

i. e.,

$$u = \text{const}\,K^{-\frac{1}{2}}\exp\left\{-\int^{s}\tfrac{1}{2}\operatorname{div}\boldsymbol{e}_s\,ds\right\},$$

the path of integration here being along a ray emerging from the point P. From this expression for u, and from (B-8) and (B-9), we find that the function

$$H(P,P') = \left(\int_P^{P_1} ds\right)^{-1}\exp\left\{-\int_{P_1}^{P'}\tfrac{1}{2}\operatorname{div}\boldsymbol{e}_s\,ds\right\}\sqrt{\frac{K(P)}{K(P')}}\exp\left\{-\int_P^{P'}K\,ds\right\} \tag{B-10}$$

is an approximate solution of (B-1). Here, the point P is considered to be fixed and the point P' to be variable, and the integrations are performed along the ray joining P and P' (see Fig. 7); P_1 is an arbitrary point on this ray (conveniently chosen to lie close to P). Because all the rays go through the same point P, a simple geometrical consideration shows that

$$\underset{\text{(at the point } P')}{\tfrac{1}{2}\operatorname{div}\boldsymbol{e}_s} \quad\rightarrow\quad \frac{1}{|\boldsymbol{r}_{P'}-\boldsymbol{r}_P|} \qquad \text{as } P'\rightarrow P.$$

If P' lies sufficiently close to P (and if P_1 is chosen to lie close to P), the formula (B-10) therefore becomes

$$H(P,P') = \frac{1}{|\boldsymbol{r}_{P'}-\boldsymbol{r}_P|}\sqrt{\frac{K(P)}{K(P')}}\exp\left\{-\int_P^{P'}K(\boldsymbol{r})\,ds\right\}.$$

Hence,

$$H(P,P') \quad\rightarrow\quad \frac{1}{|\boldsymbol{r}_{P'}-\boldsymbol{r}_P|} \qquad \text{as } P'\rightarrow P.$$

For the validity of the approximate solution (B-10) it is sufficient that simultaneously

$$\left|\frac{1}{K}\Delta\frac{1}{K}\right| << \left|\operatorname{grad}\frac{1}{K}\right| << 1$$

$$\left|\frac{1}{K}\operatorname{grad}\frac{\operatorname{div}\boldsymbol{e}_s}{K}\right| << \left|\frac{\operatorname{div}\boldsymbol{e}_s}{K}\right| << 1\,.$$

The first series of these inequalities is the same as the condition for the validity of the second order approximation of the one-dimensional *WKB*-method, whereas the second series of the above inequalities appears only in the three-dimensional *WKB*-method. As P' approaches P the inequality $|\operatorname{div}\boldsymbol{e}_s| << K$ becomes much violated; nevertheles, (B-10) may still remain an approximate solution of (B-1) and, hence, the

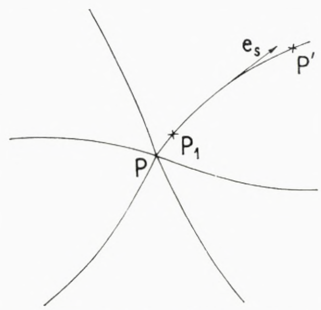

Fig. 7. The figure shows rays emerging from the point P. On the ray PP', we have indicated the point P_1 and the unit vector $\boldsymbol{e}_S$ along the ray in an arbitrary point on it.

singularity which we have found for $H(P, P')$ may be relevant. Thus, a consideration of a spherically symmetric potential barrier indicates that for this case the approximate solution (B-10) is valid in the immediate neighbourhood of the point P if $\left|\operatorname{grad}\frac{1}{K}\right| << 1$.

The function $H(P, P')$ which we have constructed and discussed above will appear as the kernel in the approximate integral representation of the alpha wave function which we now derive.

Considered as a function of P' the function $H(P, P')$, defined by (B-10), is a second order *WKB*-approximation solution of (B-1) having the singularity $1/|\boldsymbol{r}_{P'} - \boldsymbol{r}_P|$ at $\boldsymbol{r}_{P'} = \boldsymbol{r}_P$. The alpha wave function $\psi(P')$ inside the potential barrier is also a solution of (B-1), and therefore we get*

$$\psi(P) = -\frac{1}{4\pi}\iint\left\{\psi(P')\operatorname{grad}_{P'}H(P,P') - H(P,P')\operatorname{grad}_{P'}\psi(P')\right\}\cdot d\boldsymbol{\sigma}_{P'} \quad \text{(B-11)}$$

from Green's formula, the surface element at P' being denoted by $d\sigma_{P'}$. In the formula (B-11), which is analogous to Kirchhoff's formula in optics, we consider the inte-

* The formula (B-11) is exact as far as the kernel $H(P, P')$ with the required singularity at $\boldsymbol{r}_{P'} = \boldsymbol{r}_P$ is a correct solution of (B-1). As has been pointed out by Groenewold[32], it is possible in principle to use an exact formula for this kernel.

gration to be performed over the nuclear surface and over the surface of a sphere concentric with the nuclear surface and conveniently chosen somewhere in the potential barrier (see Fig. 8). Since the alpha wave function ψ on this outer surface of integration is small compared to the same wave function on the nuclear surface, and since $H(P, P')$ contains the factor $\exp\left\{-\int_P^{P'} K\,ds\right\}$, the integral over the outer surface is negligible compared to the integral over the nuclear surface, unless the point P lies in the vicinity of the outer surface. In order to obtain the alpha wave

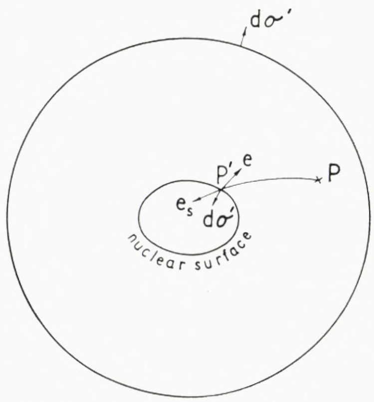

Fig. 8. Geometrical illustration of the quantities appearing in (B-11) and (B-14). The integrals in these formulae shall be performed over the nuclear surface and over the larger spherical surface which lies somewhere in the potential barrier. The surface elements $d\sigma'$ on these two surfaces have the directions shown in the figure. Unless the point P lies in the vicinity of the outer spherical surface, the integral over this surface is negligible compared to that over the nuclear surface. The curve PP' is the ray joining the point P in the potential barrier and a point P' on the nuclear surface, and $\boldsymbol{e}_s$ is the unit vector along this ray at the point P'. The gradient of the alpha wave function at the point P' has the direction -$\boldsymbol{e}$ where $\boldsymbol{e}$ is a unit vector which we assume to be approximately orthogonal to the nuclear surface at the point P'. For the most important rays PP', the unit vectors $\boldsymbol{e}_s$ and $\boldsymbol{e}$ have approximately opposite directions.

function in the interior region of the potential barrier, where we wish to apply this approximation, it is therefore sufficient to perform the integral in (B-11) only over the nuclear surface. To simplify (B-11) further we note that, in first approximation,

$$\mathrm{grad}_{P'} H(P, P') = -K(P')\, H(P, P')\, \boldsymbol{e}_s, \tag{B-12}$$

according to (B-10). Furthermore, if ψ varies slowly on the nuclear surface compared with its rapid variation in the radial direction, we can write approximately

$$\mathrm{grad}_{P'} \psi(P') = -K(P')\, \psi(P')\, \boldsymbol{e}, \tag{B-13}$$

where $\boldsymbol{e}$ denotes a unit vector which is directed almost radially outwards from the nuclear surface. The minus sign in (B-13) is due to the fact that ψ is decaying exponentially in the outward direction. Substituting (B-12) and (B-13) into (B-11), we get the approximate formula

$$\psi(P) = \frac{1}{4\pi}\iint H(P,P')\,K(P')\,\psi(P')\,(\boldsymbol{e}_s - \boldsymbol{e})\cdot d\sigma_{P'}, \tag{B-14}$$

where the integration is to be performed over the nuclear surface. The vector $\boldsymbol{e}$ is directed essentially radially outwards from the nuclear surface, whereas $\boldsymbol{e}_s$ and $d\sigma_{P'}$ are directed essentially in the opposite direction (cf. Fig. 8). The formula (B-14), which is analogous to a formula appearing in KIRCHHOFF's treatment of the diffraction of light[18], has a similar relation to CHRISTY's approximation method as the just mentioned formula in KIRCHHOFF's theory has to HUYGHENS' principle.

To illustrate the use of (B-14) and to investigate the accuracy of this formula, we now consider the well-known case that the potential $V(\boldsymbol{r})$ is spherically symmetric. Using polar coordinates (r, ϑ, φ), we find that the extremal problem (B-5) can be written

$$\int_{r_{P'}}^{r_P} K(r)\sqrt{1 + r^2\left[\left(\frac{d\vartheta}{dr}\right)^2 + \sin^2\vartheta\left(\frac{d\varphi}{dr}\right)^2\right]}\,dr = \text{minimum}.$$

The corresponding Euler-Lagrange equations give

$$\frac{K(r)\,r^2\sin^2\vartheta\,\dfrac{d\varphi}{dr}}{\sqrt{1 + r^2\left[\left(\dfrac{d\vartheta}{dr}\right)^2 + \sin^2\vartheta\left(\dfrac{d\varphi}{dr}\right)^2\right]}} = C \tag{B-15}$$

and

$$\left(\frac{K(r)\,r^2\,\dfrac{d\vartheta}{dr}}{\sqrt{1 + r^2\left[\left(\dfrac{d\vartheta}{dr}\right)^2 + \sin^2\vartheta\left(\dfrac{d\varphi}{dr}\right)^2\right]}}\right)^2 + \frac{C^2}{\sin^2\vartheta} = D^2, \tag{B-16}$$

C and D being integration constants. From (B-15) and (B-16) we get

$$-\frac{ds}{dr} = \sqrt{1 + r^2\left[\left(\frac{d\varphi}{dr}\right)^2 + \sin^2\vartheta\left(\frac{d\varphi}{dr}\right)^2\right]} = \left\{1 - \left(\frac{D}{r\,K(r)}\right)^2\right\}^{-\frac{1}{2}} \approx 1 + \tfrac{1}{2}\left(\frac{D}{r\,K(r)}\right)^2. \tag{B-17}$$

On a ray emerging from the point P we consider a variable point P'. The coordinates of P' with respect to a polar coordinate system with the axis passing through P are denoted by $(r_{P'}, \Delta\vartheta, \Delta\varphi)$. In the spherically symmetric case, the rays must obviously be plane curves, and therefore $\Delta\varphi$ is constant along any ray through P. Using the coordinates $(r_{P'}, \Delta\vartheta, \Delta\varphi)$ instead of (r, ϑ, φ) in (B-15), (B-16), and (B-17), we therefore find that $C = 0$ and

$$\frac{d\Delta\vartheta}{dr_{P'}} = -\frac{D}{r_{P'}^2 K(r_{P'})\sqrt{1 - \left(\dfrac{D}{r_{P'}K(r_{P'})}\right)^2}} \approx -\frac{D}{r_{P'}^2 K(r_{P'})}, \tag{B-18}$$

and, hence,

$$D = \frac{\Delta\vartheta}{\displaystyle\int_{r_{P'}}^{r_P} \frac{dr}{r^2 K(r)\sqrt{1-\left(\frac{D}{r K(r)}\right)^2}}} \approx \frac{\Delta\vartheta}{\displaystyle\int_{r_{P'}}^{r_P} \frac{dr}{r^2 K(r)}}. \tag{B-19}$$

The signs on the right-hand sides of (B-18) and (B-19) apply to the case that $r_{P'} < r_P$, if D is chosen to be positive. We shall restrict ourselves to this case, but we remark that the signs are to be reversed if $r_{P'} > r_P$. By substituting (B-19) into (B-17), we get

$$-\frac{ds}{dr} = \sqrt{1 + r^2\left[\left(\frac{d\vartheta}{dr}\right)^2 + \sin^2\vartheta\left(\frac{d\varphi}{dr}\right)^2\right]} \approx 1 + \frac{1}{2}\left(\frac{\Delta\vartheta}{r K(r)\displaystyle\int_{r_{P'}}^{r_P}\frac{dr}{r^2 K(r)}}\right)^2 \tag{B-20}$$

and hence, approximately

$$\int_P^{P'} K\, ds = \int_{r_{P'}}^{r_P} K(r)\, dr + \frac{(\Delta\vartheta)^2}{2c}, \tag{B-21}$$

where

$$c = \int_{r_{P'}}^{r_P} \frac{dr}{r^2 K(r)}. \tag{B-22}$$

From (B-10) and (B-21) it is clear that, in the integration over P' in (B-14), we get essential contributions only when

$$\frac{(\Delta\vartheta)^2}{2c} \lesssim 1.$$

This justifies the expansions which have been made in (B-17), (B-18), and (B-19), for in typical cases

$$\left(\frac{D}{r K(r)}\right)^2 \approx \left(\frac{\Delta\vartheta}{r K(r)\displaystyle\int_{r_{P'}}^{r_P}\frac{dr}{r^2 K(r)}}\right)^2 \lesssim 0.1\,\frac{(\Delta\vartheta)^2}{2c},$$

if $r_{P'} \leqslant r \leqslant r_P$ and P' lies on the nuclear surface and P lies somewhere in the middle of the potential barrier.

Before we can proceed further we must calculate div $\boldsymbol{e}_s$ in our spherically symmetric potential barrier. To this purpose, we consider a (somewhat cone-like) surface generated by a narrow bundle of rays emerging from the point P, a variable point on any one of these rays being denoted by P'. The area which this surface (tube of rays) cuts out on the surface through P', which cuts the rays emerging from P orthogonally, will be denoted by A. For small angles $\Delta\vartheta$ the surface element A is approximately orthogonal to $\boldsymbol{r}_{P'}$. Using this fact and the formula

$$\frac{d\,(\Delta\vartheta)^2}{dr_{P'}} \approx -\frac{2\,(\Delta\vartheta)^2}{r_{P'}^2\,K\,(r_{P'})\displaystyle\int_{r_{P'}}^{r_P}\frac{dr}{r^2\,K\,(r)}}$$

which follows from (B-18) and (B-19), we get

$$\frac{dA}{dr_{P'}} \approx -2\left\{\frac{1}{r_{P'}^2\,K\,(r_{P'})\displaystyle\int_{r_{P'}}^{r_P}\frac{dr}{r^2\,K\,(r)}} - \frac{1}{r_{P'}}\right\}A\,.$$

Furthermore, according to Gauss' integral formula, we have

$$\underset{\text{(at the point } P')}{\operatorname{div}\boldsymbol{e}_s} \approx \frac{dA}{A\,ds_{P'}} \approx -\frac{1}{A}\frac{dA}{dr_{P'}}$$

and, hence,

$$\underset{\text{(at the point } P')}{\operatorname{div}\boldsymbol{e}_s} \approx 2\left\{\frac{1}{r_{P'}^2\,K\,(r_{P'})\displaystyle\int_{r_{P'}}^{r_P}\frac{dr}{r^2\,K\,(r)}} - \frac{1}{r_{P'}}\right\}, \tag{B-23}$$

if $\Delta\vartheta$ is small.

From (B-23) we easily find that

$$\left(\int_P^{P_1} ds\right)^{-1} \exp\left\{-\int_{P_1}^{P'} \tfrac{1}{2}\operatorname{div}\boldsymbol{e}_s\,ds\right\} \approx \frac{1}{r_{P'}\,r_P\,K\,(r_P)\displaystyle\int_{r_{P'}}^{r_P}\frac{dr}{r^2\,K\,(r)}}, \tag{B-24}$$

if $\Delta\vartheta$ is small and P_1 is chosen to lie very close to P. From (B-10), (B-21), and (B-24) we then get the approximate formula

$$H\,(P,\,P')\,K\,(P') = \frac{1}{c\,r_{P'}\,r_P}\sqrt{\frac{K\,(r_{P'})}{K\,(r_P)}}\exp\left\{-\int_{r_{P'}}^{r_P} K\,(r)\,dr - \frac{(\Delta\vartheta)^2}{2\,c}\right\}, \tag{B-25}$$

where c is defined by (B-22).

We now assume the wave function $\psi\,(P')$ at any point $P'\,(r_{P'},\,\vartheta',\,\varphi')$ on the spherical surface $r_{P'} = R_0$ to be equal to $Y_{l,\,m}(\vartheta',\,\varphi')$, and calculate the wave function $\psi\,(P)$ at the point $P\,(r_P,\,\vartheta,\,\varphi)$ in our spherically symmetric potential barrier. To this purpose, we substitute (B-25) into (B-14). If we then replace, firstly, $(\boldsymbol{e}_s - \boldsymbol{e})\cdot d\sigma_{P'}$ by $2\,R_0^2 \sin(\Delta\vartheta)\,d(\Delta\vartheta)\,d(\Delta\varphi)$, secondly, $(\Delta\vartheta)^2$ by $2\{1-\cos(\Delta\vartheta)\}$, thirdly, $\cos(\Delta\vartheta)$ by z and use the formula

$$Y_{l,\,m}(\vartheta',\,\varphi') = \sum_n D^l_{m,\,n}(\vartheta,\,\varphi,\,\ldots)\;Y_{l,\,n}(\Delta\vartheta,\,\Delta\varphi),$$

we get the approximate result

$$\psi(P) = \frac{R_0}{r_P}\sqrt{\frac{K(R_0)}{K(r_P)}}\exp\left\{-\int_{R_0}^{r_P} K(r)\,dr\right\}\alpha_l\,Y_{l,m}(\vartheta,\varphi), \tag{B-26}$$

where*

$$\alpha_l = \frac{1}{c}\int_{-1}^{+1} P_l(z)\,e^{\frac{z-1}{c}}\,dz = \frac{2}{c}\,e^{-\frac{1}{c}}\,i^l j_l\left(-\frac{i}{c}\right), \tag{B-27}$$

j_l denoting a spherical Bessel function. However, this result cannot be used for very large l-values, since the formula (B-14), on which the calculations leading to (B-26) and (B-27) are based, is valid only if the wave function varies slowly on the nuclear surface compared with its rapid variation in the radial direction.

The coefficient α_l is real and positive, since it is seen, e.g., from the power series representing the spherical Bessel function $j_l(z)$ that $i^l j_l(-ix)$ is real and positive if $x > 0$. For the calculation of α_l, according to (B-27), without introducing any further approximations, one can start from the formulae

$$\begin{cases} \alpha_0 = 1 - e^{-\frac{2}{c}} \\ \alpha_1 = (1-c) + (1+c)\,e^{-\frac{2}{c}} \end{cases}$$

and use the recursion formula

$$\alpha_l = \alpha_{l-2} - (2\,l-1)\,c\,\alpha_{l-1}.$$

For an approximate evaluation of (B-27) one can use the formula

$$\left.\begin{aligned} i^l j_l(-ix) &\approx \frac{1}{2\,x}\exp\left\{\int^x \sqrt{1+\frac{l(l+1)}{x^2}}\,dx\right\} \\ &= \frac{1}{2\,x}\exp\left\{\sqrt{l(l+1)}\left[\sqrt{\frac{x^2}{l(l+1)}+1} - \ln\left(\sqrt{\frac{l(l+1)}{x^2}} + \sqrt{1+\frac{l(l+1)}{x^2}}\right)\right]\right\}, \end{aligned}\right\} \tag{B-28}$$

which is obtained by applying a first order *WKB*-approximation to the differential equation for the function $x\,j_l(-ix)$, and which is valid if $x >> 1$. If the more restrictive condition $\{l(l+1)\}^2/x^3 << 1$ is fulfilled, the formula (B-28) simplifies to

$$i^l j_l(-ix) \approx \frac{1}{2\,x}\exp\left\{x - \frac{l(l+1)}{2\,x}\right\}, \tag{B-29}$$

and from this formula it follows that α_l is approximately equal to $\exp\left\{-\frac{1}{2}\,cl\,(l+1)\right\}$ if $c^3\left\{l\,(l+1)\right\}^2 << 1$. The results of some numerical calculations of α_l in the different degrees of approximation now discussed are shown in Fig. 9.

* Christy has also applied his approximation method to the spherically symmetric penetration problem and found that the alpha wave function corresponding to the angular momentum l contains the l-dependent factor α_l given by (B-27) (private communication).

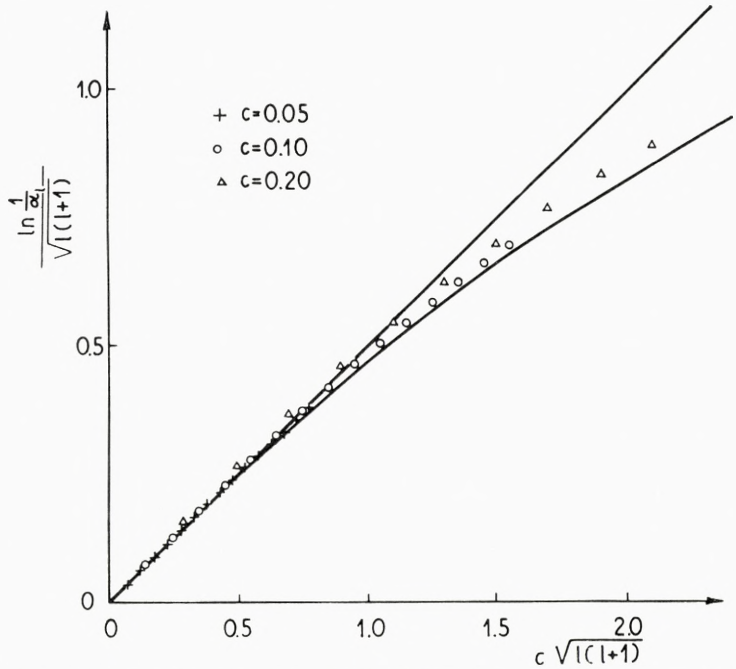

Fig. 9. Results of some numerical calculations of α_l in different degrees of approximation. The points denoted by crosses, circles and triangles (which correspond to $c = 0.05$, $c = 0.10$, and $c = 0.20$, respectively) have been obtained from the formula (B-27) without introducing any further approximations. The curved line has been obtained from (B-27) and the approximate formula (B-28), whereas the straight line has been found from (B-27) and the formula (B-29), which is still more approximate than (B-28). In the approximation represented by the straight line, α_l is given by the well-known expression $\exp\left\{-\frac{1}{2}\, cl\,(l+1)\right\}$.

In the usual treatment of the penetration of an alpha particle of given angular momentum l through a spherically symmetric Coulomb potential $U_0(r)$, this problem is reduced to the problem of finding the appropriate solution $G_l(E, r)$ of the differential equation (A-1) in Appendix A. If the alpha wave function on the spherical surface $r_{P'} = R_0$ is assumed to be equal to $Y_{l,m}(\vartheta', \varphi')$, the alpha wave function at a point P inside the potential barrier contains an l-dependent factor corresponding to α_l in (B-26) and being the quotient of $G_l(E, r_P)/G_l(E, R_0)$ and $G_0(E, r_P)/G_0(E, R_0)$. The value which is obtained for this quotient when the strict WKB-approximation (A-13 a) and (A-14 a) in Appendix A is used for the functions $G_l(E, r)$ and $G_0(E, r)$ will be denoted by γ_l. In the limit of small values of $l(l+1)/\varkappa^2$, the formulae (A-13 a) and (A-14 a) can be replaced by (A-15 a), and then the above mentioned quotient reduces to a quantity which will be denoted by β_l, and which is given by

$$\beta_l = e^{-\frac{1}{2} cl\,(l+1)}, \qquad \text{(B-30)}$$

where

$$c = \frac{2}{\varkappa}\left(\sqrt{\frac{\varkappa}{k R_0} - 1} - \sqrt{\frac{\varkappa}{k r_P} - 1}\right) = \int_{R_0}^{r_P} \frac{dr}{r^2 K(r)}. \tag{B-31}$$

The quantities α_l, β_l, and γ_l have the same physical meaning, but they are obtained by different approximation methods. In Tables 14 and 15, the results of some numerical comparisons between these three quantities are shown. It is seen that α_l, β_l, and γ_l are approximately equal up to fairly large values of l.

TABLE 14.
Numerical values of α_l/β_l for different values of c.

l	α_l/β_l for $c = 0.05$	for $c = 0.10$	for $c = 0.20$
0	1	1	1
1	0.999	0.995	0.977
2	0.996	0.986	0.947
3	0.993	0.974	0.930
4	0.989	0.966	0.944
5	0.986	0.964	1.008
6	0.983	0.971	1.152
7	0.981	0.994	1.427
8	0.982	1.036	1.94
9	0.987	1.106	
10	0.995	1.21	
11	1.009	1.37	
12	1.029	1.61	
13	1.057	1.96	
14	1.095	2.5	
15	1.145		

TABLE 15.
Numerical values of α_l/β_l and β_l/γ_l for $kr_{P'}/\varkappa = 0.2$, $kr_P/\varkappa = 0.6$ and $\varkappa = 50$.

l	α_l/β_l	β_l/γ_l
0	1	1
1	0.999	1.000
2	0.997	0.998
3	0.994	0.996
4	0.990	0.992
5	0.987	0.985
6	0.984	0.974
7	0.983	0.957
8	0.983	0.935
9	0.986	0.905
10	0.993	0.869
11	1.004	0.823
12	1.020	0.770
13	1.043	0.709
14	1.075	0.642
15	1.116	0.570

After this detailed treatment of the spherically symmetric penetration problem, we now consider the penetration of an alpha particle through the anisotropic potential $V(\boldsymbol{r})$ outside the surface of a deformed nucleus which we suppose to be fixed in space. We write the equation for the nuclear surface in the general form

$$r = R(\vartheta'), \tag{B-32}$$

and we assume that the alpha wave function on this surface is a given function $\psi_0(\vartheta', \varphi')$. Assuming the purely anisotropic part $V(\boldsymbol{r}) - U_0(r)$ of the potential $V(\boldsymbol{r})$ to be small compared with its spherically symmetric part $U_0(r)$, we can in first approximation use the same rays as for the spherically symmetric potential $U_0(r)$. The reason is that the ray between any two given points P and P' in the potential barrier has the extremal property (B-5), which implies that the integral $\int_P^{P'} K\, ds$ along the ray between these two points is stationary for small variations of the path of integration connecting P and P'. According to (B-20), we then have

$$-\frac{ds}{dr} \approx 1 + \frac{1}{2}\left(\frac{\Delta\vartheta}{r\,K_0(r)\int_{r_{P'}}^{r_P}\frac{dr}{r^2 K_0(r)}}\right)^2$$

where $\Delta\vartheta$ is the angle (viewed from the center of the nucleus) between a point P in the potential barrier and a point P' on the nuclear surface, and $K_0(r)$ is the function which we obtain from $K(\boldsymbol{r})$ by replacing $V(\boldsymbol{r})$ by $U_0(r)$. Hence, we get the approximate formula

$$\int_P^{P'} K\,ds = \int_{r_{P'}}^{r_P} K\,dr + \frac{(\Delta\vartheta)^2}{2\int_{r_{P'}}^{r_P}\frac{dr}{r^2 K}}$$

if we neglect terms of the order of magnitude $(\Delta K/K)^2$, where $\Delta K = K(\boldsymbol{r}) - K_0(r)$. This formula for $\int_P^{P'} K\,ds$ can be written

$$\int_P^{P'} K\,ds = \int_{R_0}^{r_P} K_0(r)\,dr + \int_{R_0}^{r_P} \Delta K(r,\vartheta)\,dr - \int_{R_0}^{r_{P'}} K(r,\vartheta)\,dr + \frac{(\Delta\vartheta)^2}{2\int_{r_{P'}}^{r_P}\frac{dr}{r^2 K(r,\vartheta)}},$$

where R_0 is a constant conveniently chosen to be some average value of $R(\vartheta')$. The angle ϑ is here a function of r which is geometrically represented by the ray passing through the points P and P'. In the last term it is safe to replace $r_{P'}$ by R_0 and $K(r,\vartheta)$ by $K_0(r)$. In the third term, we can approximately replace the variable angle ϑ by the angle ϑ' for the point P', for $|r_{P'} - R_0|$ is small compared to R_0, and the most important rays go essentially radially in the neighbourhood of the nuclear surface. The third term turns out to be more important than the second one, and therefore we may evaluate the latter in a rather approximate way. Since ΔK decreases rapidly with increasing r, the essential contributions to the integral $\int_{R_0}^{r_P} \Delta K\,dr$ are obtained for small values of r, and therefore we approximately replace the variable angle ϑ appearing in the integrand $\Delta K(r,\vartheta)$ by the angle ϑ' for the point P'. Substituting the resulting formula for $\int_P^{P'} K\,ds$ into (B-10), and assuming that P lies so far away from the nuclear surface that the variation in $H(P,P')$ as P' changes slightly is due essentially only to the change of the integral $\int_P^{P'} K\,ds$ in the exponent of (B-10), we get the approximate formula

$$H(P, P') = H_0(P, P_0) \exp\left\{\int_{R_0}^{R(\vartheta')} K(r, \vartheta')\, dr - \int_{R_0}^{r_P} \Delta K(r, \vartheta')\, dr\right\}, \tag{B-33}$$

where P_0 is the point on the sphere $r = R_0$ which has the same polar angles (ϑ', φ') as the point P' on the nuclear surface, and the index 0 on the kernel $H_0(P, P_0)$ indicates that this kernel refers to the case that the potential is spherically symmetric and equal to $U_0(r)$. From (B-14) and (B-33) it follows that, in first approximation, the alpha wave function at the point P in the anisotropic potential barrier $V(\boldsymbol{r})$ is the same as if the electrostatic potential were equal to $U_0(r)$ (= the spherically symmetric part of $V(\boldsymbol{r})$) and the alpha wave function ψ on the sphere $r = R_0$ were equal to

$$\psi_0(\vartheta', \varphi') \exp\left\{\int_{R_0}^{R(\vartheta')} K(r, \vartheta')\, dr - \int_{R_0}^{r_P} \Delta K(r, \vartheta')\, dr\right\}. \tag{B-34}$$

This approximate result shows clearly how the alpha wave function is affected in a very simple and natural way by the variation in thickness of the potential barrier due to the shape of the nucleus and by the variation in height of the anisotropic potential. After having resolved the expression (B-34) in terms of spherical harmonics, one can immediately give the alpha wave function in the potential barrier as a superposition of expressions such as that on the right-hand side of (B-26), which we have previously shown to be essentially equivalent to

$$\frac{R_0}{r_P} \frac{G_l(E, r_P)}{G_l(E, R_0)} Y_{l,m}(\vartheta, \varphi).$$

Hence, we get the final approximate formula

$$\left.\begin{aligned} \psi(r, \vartheta, \varphi) &= \sum_l \sum_\Omega \frac{R_0}{r} \frac{G_l(E, r)}{G_l(E, R_0)} Y_{l,\Omega}(\vartheta, \varphi) \\ \times \int_0^\pi \int_0^{2\pi} \psi_0(\vartheta', \varphi') & \exp\left\{\int_{R_0}^{R(\vartheta')} K(r, \vartheta')\, dr - \int_{R_0}^{r} \Delta K(r, \vartheta')\, dr\right\} Y^*_{l,\Omega}(\vartheta', \varphi') \sin\vartheta'\, d\vartheta'\, d\varphi' \end{aligned}\right\} \tag{B-35}$$

for the alpha wave function in our anisotropic potential barrier.

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